A unifying type-theory for higher-order (amortized) cost analysis
Technical Report

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Contents

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1 Development for dlPCF's embedding

1.1 Syntax

Expressions	e	$::=$	$v \mid e_1\ e_2 \mid \langle\langle e_1, e_2\rangle\rangle \mid \mathsf{let}\langle\langle x, y\rangle\rangle = e_1 \mathsf{\ in\ } e_2 \mid$ $\langle e, e\rangle \mid \mathsf{fst}(e) \mid \mathsf{snd}(e) \mid \mathsf{inl}(e) \mid \mathsf{inr}(e) \mid \mathsf{case}\ e \mathsf{\ of\ } e; e \mid$ $\mathsf{let}\,!\,x = e_1 \mathsf{\ in\ } e_2 \mid e\,[] \mid e :: e \mid e; x.e$
Values	v	$::=$	$x \mid c \mid \lambda x.e \mid \langle\langle v_1, v_2\rangle\rangle \mid \langle v, v\rangle \mid \mathsf{inl}(e) \mid \mathsf{inr}(e) \mid nil \mid$ $!e \mid \Lambda.e \mid \mathsf{ret}\, e \mid \mathsf{bind}\, x = e_1 \mathsf{\ in\ } e_2 \mid \uparrow^I \mid \mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2 \mid \mathsf{store}\, e$ (No value forms for $[I]\,\tau$)
Index	I	$::=$	$N \mid i \mid I + I \mid I - I \mid \sum_{i<I} I \mid \bigtriangleup_{a}^{I,I} I \mid \lambda_s i.I \mid I\ I$
Sort	S	$::=$	$\mathbb{N} \mid \mathbb{R}^+ \mid S \to S$
Kind	K	$::=$	$Type \mid S \to K$
Types	τ	$::=$	$\mathbf{1} \mid \mathsf{b} \mid \tau_1 \multimap \tau_2 \mid \tau_1 \otimes \tau_2 \mid \tau_1 \,\&\, \tau_2 \mid \tau_1 \oplus \tau_2 \mid !_{a<I}\tau \mid [I]\,\tau \mid \mathbb{M}\, I\, \tau \mid$ $\alpha \mid \forall\alpha : K . \tau \mid \forall i : S . \tau \mid \lambda_t i.\tau \mid \tau\ I \mid L^I \tau \mid \exists i : S . \tau \mid c \Rightarrow \tau \mid c\&\tau$
Constraints	c	$::=$	$I = I \mid I < I \mid c \wedge c$
Lin. context for term variables	Γ	$::=$	$. \mid \Gamma, x : \tau$
Bounded Lin. context for term variables	Ω	$::=$	$. \mid \Omega, x :_{a<I} \tau$
Unres. context for sort variables	Θ	$::=$	$. \mid \Theta, i : S$
Unres. context for type variables	Ψ	$::=$	$. \mid \Psi, \alpha : K$

Definition 1 (Bounded sum of context for dlPCF)**.** $\sum_{a<I} . = .$
$\sum_{a<I} \Gamma, x : [b < J]\tau = (\sum_{a<I} \Gamma), x : [c < \sum_{a<I} J]\sigma$
where
$\tau = \sigma[(\sum_{d<a} J[d/a] + b)/c]$

Definition 2 (Bounded sum of multiplicity context)**.** $\sum_{a<I} . = .$
$\sum_{a<I} \Omega, x :_{b<J} \tau = (\sum_{a<I} \Omega), x :_{c<\sum_{a<I} J} \sigma$
where
$\tau = \sigma[(\sum_{d<a} J[d/a] + b)/c]$

Definition 3 (Binary sum of context for dlPCF)**.**

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma_1' \oplus \Gamma_2/x), x : [c < I + J]\tau & \Gamma_1 = \Gamma_1', x : [a < I]\tau[a/c] \wedge (x : [b < J]\tau[I + b/c]) \in \Gamma_2 \\ (\Gamma_1' \oplus \Gamma_2), x :_{a<I} \tau & \Gamma_1 = \Gamma_1', x : [a < I]\tau \wedge (x : [-]-) \notin \Gamma_2 \end{cases}$$

Definition 4 (Binary sum of multiplicity context)**.**

$$\Omega_1 \oplus \Omega_2 \triangleq \begin{cases} \Omega_2 & \Omega_1 = . \\ (\Omega_1' \oplus \Omega_2/x), x :_{c<I+J} \tau & \Omega_1 = \Omega_1', x :_{a<I} \tau[a/c] \wedge (x :_{b<J} \tau[I + b/c]) \in \Omega_2 \\ (\Omega_1' \oplus \Omega_2), x :_{a<I} \tau & \Omega_1 = \Omega_1', x :_{a<I} \tau \wedge (x :_{-} -) \notin \Omega_2 \end{cases}$$

Definition 5 (Binary sum of affine context)**.**

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma_1' \oplus \Gamma_2), x : \tau & \Gamma_1 = \Gamma_1', x : \tau \wedge (x : -) \notin \Gamma_2 \end{cases}$$

1.2 Typesystem

$\boxed{\text{Typing } \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}$

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau}\text{ T-var1} \qquad \frac{\Theta, \Delta \models I \geqslant 1}{\Psi; \Theta; \Delta; \Omega, x :_{a<I} \tau; \Gamma \vdash x : \tau[0/a]}\text{ T-var2}$$

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}}\text{ T-unit} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathsf{b}}\text{ T-base}$$

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash nil : L^0\ \tau}\text{ T-nil}$$

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : L^n\tau \qquad \Theta; \Delta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1}\tau}\text{ T-cons}$$

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : L^n\ \tau \\ \Psi; \Theta; \Delta, n = 0; \Omega_2; \Gamma_2 \vdash e_1 : \tau' \qquad \Psi; \Theta; \Delta, n > 0; \Omega_2; \Gamma_2, h : \tau, t : L^{n-1}\tau \vdash e_2 : \tau'\end{array}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2 : \tau'}\text{ T-match}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \qquad \Theta; \Delta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S\ .\tau}\text{ T-existI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s : S\ .\tau \qquad \Psi; \Theta, s : S; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'}\text{ T-existE}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x.e : (\tau_1 \multimap \tau_2)}\text{ T-lam}$$

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1\ e_2 : \tau_2}\text{ T-app}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'}\text{ T-sub}$$

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \langle\!\langle e_1, e_2 \rangle\!\rangle : (\tau_1 \otimes \tau_2)}\text{ T-tensorI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\langle\!\langle x, y \rangle\!\rangle = e\ \mathsf{in}\ e' : \tau}\text{ T-tensorE}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \,\&\, \tau_2)}\text{ T-withI} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \,\&\, \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{fst}(e) : \tau_1}\text{ T-fst}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \,\&\, \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{snd}(e) : \tau_2}\text{ T-snd}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau_1}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \mathsf{inl}(e) : \tau_1 \oplus \tau_2}\ \text{T-inl} \qquad \frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau_2}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \mathsf{inr}(e) : \tau_1 \oplus \tau_2}\ \text{T-inr}$$

$$\frac{\begin{array}{c}\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \\ \Psi;\Theta;\Delta;\Omega_2;\Gamma_2, x : \tau_1 \vdash e_1 : \tau \qquad \Psi;\Theta;\Delta;\Omega_2;\Gamma_2, y : \tau_2 \vdash e_2 : \tau\end{array}}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1; e_2 : \tau}\ \text{T-case}$$

$$\frac{\Psi;\Theta, a;\Delta, a < I;\Omega;. \vdash e : \tau}{\Psi;\Theta;\Delta;\sum_{a<I}\Omega;. \vdash !e : !_{a<I}\tau}\ \text{T-subExpI}$$

$$\frac{\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e : (!_{a<I}\tau) \qquad \Psi;\Theta;\Delta;\Omega_2, x :_{a<I} \tau;\Gamma_2 \vdash e' : \tau'}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\,!\,x = e\ \mathsf{in}\ e' : \tau'}\ \text{T-subExpE}$$

$$\frac{\Psi, \alpha : K;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \Lambda.e : (\forall \alpha : K .\tau)}\ \text{T-tabs}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : (\forall \alpha : K .\tau) \qquad \Psi;\Theta;\Delta \vdash \tau' : K}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e\,[] : (\tau[\tau'/\alpha])}\ \text{T-tapp}$$

$$\frac{\Psi;\Theta, i : S;\Delta;\Omega;\Gamma \vdash e : \tau}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \Lambda.e : (\forall i : S .\tau)}\ \text{T-iabs} \qquad \frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : (\forall i : S .\tau) \qquad \Theta;\Delta \vdash I : S}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e\,[] : (\tau[I/i])}\ \text{T-iapp}$$

$$\frac{\Psi;\Theta, b;\Delta, b < L;\Omega, x :_{a<I} \tau[(b + 1 + \bigtriangleup_{b}^{b+1,a} I)/b];. \vdash e : \tau \qquad L \geqslant \bigtriangleup_{b}^{0,1} I}{\Psi;\Theta;\Delta;\sum_{b<L}\Omega;. \vdash \mathsf{fix}x.e : \tau[0/b]}\ \text{T-fix}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau \qquad \Psi;\Theta;\Delta \models \Gamma' \sqsubseteq \Gamma \qquad \Psi;\Theta;\Delta \models \Omega' \sqsubseteq \Omega}{\Psi;\Theta;\Delta;\Omega';\Gamma' \vdash e : \tau}\ \text{T-weaken}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \mathsf{ret}\,e : \mathbb{M}\,0\,\tau}\ \text{T-ret}$$

$$\frac{\begin{array}{c}\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e_1 : \mathbb{M}\,I_1\,\tau_1 \\ \Psi;\Theta;\Delta;\Omega_2;\Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}\,I_2\,\tau_2 \qquad \Theta;\Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta;\Delta \vdash I_2 : \mathbb{R}^+\end{array}}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \mathsf{bind}\,x = e_1\ \mathsf{in}\ e_2 : \mathbb{M}(I_1 + I_2)\,\tau_2}\ \text{T-bind}$$

$$\frac{\Theta;\Delta \vdash I : \mathbb{R}^+}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \uparrow^I : \mathbb{M}\,I\,\mathbf{1}}\ \text{T-tick}$$

$$\frac{\begin{array}{c}\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e_1 : [I_1]\,\tau_1 \\ \Psi;\Theta;\Delta;\Omega_2;\Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(I_1 + I_2)\,\tau_2 \qquad \Theta;\Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta;\Delta \vdash I_2 : \mathbb{R}^+\end{array}}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \mathsf{release}\,x = e_1\ \mathsf{in}\ e_2 : \mathbb{M}\,I_2\,\tau_2}\ \text{T-release}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau \qquad \Theta;\Delta \vdash I : \mathbb{R}^+}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \mathsf{store}\, e : \mathbb{M}\, I\, ([I]\, \tau)} \text{ T-store} \qquad \frac{\Psi;\Theta;\Delta, c;\Omega;\Gamma \vdash e : \tau}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \Lambda.\ e : (c \Rightarrow \tau)} \text{ T-CI}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : (c \Rightarrow \tau) \qquad \Theta;\Delta \models c}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e[] : \tau} \text{ T-CE} \qquad \frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau \qquad \Theta;\Delta \models c}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : (c \& \tau)} \text{ T-CAndI}$$

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : (c \& \tau) \qquad \Psi;\Theta;\Delta, c;\Omega;\Gamma, x : \tau \vdash e' : \tau'}{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash \mathsf{clet}\, x = e\ \mathsf{in}\ e' : \tau'} \text{ T-CAndE}$$

Figure 1: Typing rules for λ-Amor

$$\frac{}{\Psi;\Theta;\Delta \vdash \tau <: \tau} \text{ sub-refl} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau_1' <: \tau_1 \qquad \Psi;\Theta;\Delta \vdash \tau_2 <: \tau_2'}{\Psi;\Theta;\Delta \vdash \tau_1 \multimap \tau_2 <: \tau_1' \multimap \tau_2'} \text{ sub-arrow}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_1' \qquad \Psi;\Theta;\Delta \vdash \tau_2 <: \tau_2'}{\Psi;\Theta;\Delta \vdash \tau_1 \otimes \tau_2 <: \tau_1' \otimes \tau_2'} \text{ sub-tensor}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_1' \qquad \Psi;\Theta;\Delta \vdash \tau_2 <: \tau_2'}{\Psi;\Theta;\Delta \vdash \tau_1 \,\&\, \tau_2 <: \tau_1' \,\&\, \tau_2'} \text{ sub-with}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_1' \qquad \Psi;\Theta;\Delta \vdash \tau_2 <: \tau_2'}{\Psi;\Theta;\Delta \vdash \tau_1 \oplus \tau_2 <: \tau_1' \oplus \tau_2'} \text{ sub-sum}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau <: \tau' \qquad \Theta;\Delta \models n' \leqslant n}{\Psi;\Theta;\Delta \vdash [n]\, \tau <: [n']\, \tau'} \text{ sub-potential}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau <: \tau' \qquad \Theta;\Delta \models n \leqslant n'}{\Psi;\Theta;\Delta \vdash \mathbb{M}\, n\, \tau <: \mathbb{M}\, n'\, \tau'} \text{ sub-monad}$$

$$\frac{\Psi;\Theta, a;\Delta, a < J \vdash \tau <: \tau' \qquad \Theta, a;\Delta \models J \leqslant I}{\Psi;\Theta;\Delta \vdash !_{a<I}\tau <: !_{a<J}\tau'} \text{ sub-subExp}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau <: \tau'}{\Psi;\Theta;\Delta \vdash L^n\ \tau <: L^n\ \tau'} \text{ sub-list} \qquad \frac{\Psi;\Theta, s;\Delta \vdash \tau <: \tau'}{\Psi;\Theta;\Delta \vdash \exists s.\tau <: \exists s.\tau'} \text{ sub-exist}$$

$$\frac{\Psi, \alpha : K;\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_2}{\Psi;\Theta;\Delta \vdash \forall \alpha : K\ .\tau_1 <: \forall \alpha.\tau_2} \text{ sub-typePoly} \qquad \frac{\Psi;\Theta, i : S;\Delta \vdash \tau_1 <: \tau_2}{\Psi;\Theta;\Delta \vdash \forall i : S\ .\tau_1 <: \forall i.\tau_2} \text{ sub-indexPoly}$$

$$\frac{\Psi;\Theta;\Delta\vdash\tau_1 <: \tau_2 \qquad \Theta;\Delta\models c_2 \implies c_1}{\Psi;\Theta;\Delta\vdash c_1\Rightarrow\tau_1 <: c_2\Rightarrow\tau_2}\text{ sub-constraint}$$

$$\frac{\Psi;\Theta;\Delta\vdash\tau_1 <: \tau_2 \qquad \Theta;\Delta\models c_1 \implies c_2}{\Psi;\Theta;\Delta\vdash c_1\&\tau_1 <: c_2\&\tau_2}\text{ sub-CAnd}$$

$$\frac{\Theta;\Delta\vdash k:\mathbb{R}^+ \qquad \Theta;\Delta\vdash k':\mathbb{R}^+}{\Psi;\Theta;\Delta\vdash [k](\tau_1\multimap\tau_2) <: ([k']\,\tau_1\multimap[k'+k]\,\tau_2)}\text{ sub-potArrow}$$

$$\frac{}{\Psi;\Theta;\Delta\vdash\tau <: [0]\,\tau}\text{ sub-potZero} \qquad \frac{\Psi;\Theta,i:S;\Delta\vdash\tau <: \tau'}{\Psi;\Theta;\Delta\vdash\lambda_s i:S.\tau <: \lambda_t i:S.\tau'}\text{ sub-familyAbs}$$

$$\frac{\Theta\vdash I:S}{\Psi;\Theta;\Delta\vdash(\lambda_t i:S.\tau)\;I <: \tau[I/i]}\text{ sub-familyApp1}$$

$$\frac{\Theta\vdash I:S}{\Psi;\Theta;\Delta\vdash\tau[I/i] <: (\lambda_t i:S.\tau)\;I}\text{ sub-familyApp2}$$

$$\frac{}{\Psi;\Theta;\Delta\vdash[\sum_{a<I}K]\,!_{a<I}\tau <: !_{a<I}\,[K]\,\tau}\text{ sub-bSum}$$

Figure 2: Subtyping

$$\frac{}{\Psi;\Theta;\Delta\vdash\Gamma\sqsubseteq .}\text{ dlpcf-subBase}$$

$$\frac{x:[a<J]\tau'\in\Gamma_1 \qquad \Psi;\Theta,a;\Delta,a<I\vdash\tau'\sqsubseteq\tau \qquad \Psi;\Theta;\Delta\vdash I\leqslant J \qquad \Psi;\Theta;\Delta\vdash\Gamma_1/x\sqsubseteq\Gamma_2}{\Theta;\Delta\vdash\Gamma_1\sqsubseteq\Gamma_2,x:[a<I]\tau}\text{ dlpcf-subInd}$$

Figure 3: Γ Subtyping for dlPCF

$$\frac{}{\Psi;\Theta;\Delta\vdash\Omega\sqsubseteq .}\text{ sub-mBase}$$

$$\frac{x:_{a<J}\tau'\in\Omega_1 \qquad \Psi;\Theta,a;\Delta,a<I\vdash\tau' <: \tau \qquad \Theta;\Delta\vdash I\leqslant J \qquad \Psi;\Theta;\Delta\vdash\Omega_1/x\sqsubseteq\Omega_2}{\Psi;\Theta;\Delta\vdash\Omega_1\sqsubseteq\Omega_2,x:_{a<I}\tau}\text{ sub-mInd}$$

Figure 4: Ω Subtyping

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma \sqsubseteq .}\ \text{sub-lBase}$$

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta; \Delta \vdash \tau' <: \tau \qquad \Psi; \Theta; \Delta \vdash \Gamma_1/x \sqsubseteq \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2, x : \tau}\ \text{sub-lBase}$$

Figure 5: Γ Subtyping

$$\frac{}{\Theta, i : S; \Delta \vdash i : S}\ \text{S-var} \qquad \frac{}{\Theta; \Delta \vdash N : \mathbb{N}}\ \text{S-nat} \qquad \frac{}{\Theta; \Delta \vdash R : \mathbb{R}^+}\ \text{S-real} \qquad \frac{\Theta; \Delta \vdash i : \mathbb{N}}{\Theta \vdash i : \mathbb{R}^+}\ \text{S-real1}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{N}}\ \text{S-add-Nat} \qquad \frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{R}^+}\ \text{S-add-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta \vdash I_2 : \mathbb{N} \qquad \Theta; \Delta \models I_1 \geqslant I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{N}}\ \text{S-minus-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+ \qquad \Theta; \Delta \models I_1 \geqslant I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{R}^+}\ \text{S-minus-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{N}}\ \text{S-bSum-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{R}^+}\ \text{S-bSum-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta \vdash I_2 : \mathbb{N} \qquad \Theta; \Delta, a : \mathbb{N} \vdash I_3 : \mathbb{N}}{\Theta \vdash \bigcirc\!\!\!\!\triangle_{a}^{I_1, I_2}\ I_3 : \mathbb{N}}\ \text{S-forest}$$

$$\frac{\Theta, i : S; \Delta \vdash I : S'}{\Theta; \Delta \vdash \lambda_s i.\ I : S \rightarrow S'}\ \text{S-family}$$

Figure 6: Typing rules for sorts

$$\frac{}{\Psi;\Theta;\Delta \vdash \mathbf{1} : Type}\ \text{K-unit} \qquad \frac{}{\Psi;\Theta;\Delta \vdash \mathsf{b} : Type}\ \text{K-base}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \multimap \tau_2 : K}\ \text{K-arrow}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \otimes \tau_2 : K}\ \text{K-tensor}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \,\&\, \tau_2 : K}\ \text{K-with} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \oplus \tau_2 : K}\ \text{K-or}$$

$$\frac{\Psi;\Theta, a : S;\Delta, a < I \vdash \tau : K \qquad \Theta \vdash I : \mathbb{N}}{\Psi;\Theta;\Delta \vdash !_{a<I}\tau : K}\ \text{K-subExp}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \vdash I : \mathbb{R}^+}{\Psi;\Theta;\Delta \vdash [I]\,\tau : K}\ \text{K-lab} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \vdash I : \mathbb{R}^+}{\Psi;\Theta;\Delta \vdash \mathbb{M}\, I\, \tau : K}\ \text{K-monad}$$

$$\frac{\Psi, \alpha : K';\Theta;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \forall \alpha.\tau : K}\ \text{K-tabs} \qquad \frac{\Psi;\Theta, i : S;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \forall i.\tau : K}\ \text{K-iabs}$$

$$\frac{\Psi;\Theta;\Delta, c \vdash \tau : K}{\Psi;\Theta;\Delta \vdash c \Rightarrow \tau : K}\ \text{K-constraint} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \models c}{\Psi;\Theta;\Delta \vdash c \& \tau : K}\ \text{K-consAnd}$$

$$\frac{\Psi;\Theta, i : S;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \lambda_t i.\tau : S \to K}\ \text{K-family} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau : S \to K \qquad \Theta;\Delta \vdash I : S}{\Psi;\Theta;\Delta \vdash \tau\ \ I : K}\ \text{K-iapp}$$

Figure 7: Kind rules for types

1.3 Semantics

$\boxed{\text{Pure reduction, } e \Downarrow_t v}$ $\boxed{\text{Forcing reduction, } e \Downarrow_t^c v}$

$$\frac{e_1 \Downarrow_{t_1} v \qquad e_2 \Downarrow_{t_2} l}{e_1 :: e_2 \Downarrow_{t_1+t_2+1} v :: l}\ \text{E-cons} \qquad \frac{e_1 \Downarrow_{t_1} nil \qquad e_2 \Downarrow_{t_2} v}{\mathsf{match}\ e_1\ \mathsf{with}\ |nil \mapsto e_2\ |h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v}\ \text{E-matchNil}$$

$$\frac{e_1 \Downarrow_{t_1} v_h :: l \qquad e_3[v_h/h][l/t] \Downarrow_{t_2} v}{\mathsf{match}\ e_1\ \mathsf{with}\ |nil \mapsto e_2\ |h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v}\ \text{E-matchCons}$$

$$\frac{e_1 \Downarrow_{t_1} v \qquad e_2[v/x] \Downarrow_{t_2} v'}{e_1; x.e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-exist} \qquad \frac{e_1 \Downarrow_{t_1} \lambda x.e' \qquad e'[e_2/x] \Downarrow_{t_2} v'}{e_1\ e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-app}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2 \Downarrow_{t_2} v_2}{\langle\langle e_1, e_2\rangle\rangle \Downarrow_{t_1+t_2+1} \langle\langle v_1, v_2\rangle\rangle}\ \text{E-TI} \qquad \frac{e \Downarrow_{t_1} \langle\langle v_1, v_2\rangle\rangle \qquad e'[v_1/x][v_2/y] \Downarrow_{t_2} v}{\mathsf{let}\langle\langle x, y\rangle\rangle = e\ \mathsf{in}\ e' \Downarrow_{t_1+t_2+1} v}\ \text{E-TE}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2 \Downarrow_{t_2} v_2}{\langle e_1, e_2\rangle \Downarrow_{t_1+t_2+1} \langle v_1, v_2\rangle}\ \text{E-WI} \qquad \frac{e \Downarrow_t \langle v_1, v_2\rangle}{\mathsf{fst}(e) \Downarrow_{t+1} v_1}\ \text{E-fst} \qquad \frac{e \Downarrow_t \langle v_1, v_2\rangle}{\mathsf{fst}(e) \Downarrow_{t+1} v_2}\ \text{E-snd}$$

$$\frac{e \Downarrow_t v}{\mathsf{inl}(e) \Downarrow_{t+1} \mathsf{inl}(v)}\ \text{E-inl} \qquad \frac{e \Downarrow_t v}{\mathsf{inr}(e) \Downarrow_{t+1} \mathsf{inr}(v)}\ \text{E-inr} \qquad \frac{e \Downarrow_{t_1} \mathsf{inl}(v) \qquad e'[v/x] \Downarrow_{t_2} v'}{\mathsf{case}\ e\ \mathsf{of}\ e'; e'' \Downarrow_{t_1+t_2+1} \mathsf{inl}(v')}\ \text{E-case1}$$

$$\frac{e \Downarrow_{t_1} \mathsf{inr}(v) \qquad e''[v/y] \Downarrow_{t_2} v''}{\mathsf{case}\ e\ \mathsf{of}\ e'; e'' \Downarrow_{t_1+t_2+1} \mathsf{inl}(v'')}\ \text{E-case2} \qquad \frac{}{!e \Downarrow_0 !e}\ \text{E-expI}$$

$$\frac{e \Downarrow_{t_1} !e'' \qquad e'[e''/x] \Downarrow_{t_2} v}{\mathsf{let}\,!\,x = e\ \mathsf{in}\ e' \Downarrow_{t_1+t_2+1} v}\ \text{E-expE} \qquad \frac{e[\mathsf{fix}x.e/x] \Downarrow_t v}{\mathsf{fix}x.e \Downarrow_{t+1} v}\ \text{E-fix}$$

$$\frac{v \in \{(), x, nil, \lambda y.e, \Lambda.e, \mathsf{ret}\, e, \mathsf{bind}\, x = e_1\ \mathsf{in}\ e_2, \uparrow^\kappa, \mathsf{release}\, x = e_1\ \mathsf{in}\ e_2, \mathsf{store}\, e\}}{v \Downarrow_0 v}\ \text{E-val}$$

$$\frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_2} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-tapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_2} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-iapp}$$

$$\frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_1} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-CE} \qquad \frac{e_1 \Downarrow_{t_1} v \qquad e_2[v/x] \Downarrow_{t_2} v'}{\mathsf{clet}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-CandE} \qquad \frac{e \Downarrow_t v}{\mathsf{ret}\, e \Downarrow^0_{t+1} v}\ \text{E-return}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad v_1 \Downarrow^{c_1}_{t_2} v_1' \qquad e_2[v_1'/x] \Downarrow_{t_3} v_2 \qquad v_2 \Downarrow^{c_2}_{t_4} v_2'}{\mathsf{bind}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow^{c_1+c_2}_{t_1+t_2+t_3+t_4+1} v_2'}\ \text{E-bind} \qquad \frac{}{\uparrow^\kappa \Downarrow^\kappa_1 ()}\ \text{E-tick}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2[v_1/x] \Downarrow_{t_2} v_2 \qquad v_2 \Downarrow^{c}_{t_3} v_2'}{\mathsf{release}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow^{c}_{t_1+t_2+t_3+1} v_2'}\ \text{E-release} \qquad \frac{e \Downarrow_t v}{\mathsf{store}\, e \Downarrow^0_{t+1} v}\ \text{E-store}$$

Figure 8: Evaluation rules: pure and forcing

1.4 Model

Definition 6 (Value and expression relation).

$$
\begin{array}{lcl}
[\![\mathbf{1}]\!] & \triangleq & \{(p, T, ())\} \\
[\![\mathsf{b}]\!] & \triangleq & \{(p, T, v) \mid v \in [\![\mathsf{b}]\!]\} \\
[\![L^0 \tau]\!] & \triangleq & \{(p, T, nil)\} \\
[\![L^{s+1} \tau]\!] & \triangleq & \{(p, T, v :: l) | \exists p_1, p_2 . p_1 + p_2 \leqslant p \wedge (p_1, T, v) \in [\![\tau]\!] \wedge (p_2, T, l) \in [\![L^s \tau]\!]\} \\
[\![\tau_1 \otimes \tau_2]\!] & \triangleq & \{(p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \mid \exists p_1, p_2 . p_1 + p_2 \leqslant p \wedge (p_1, T, v_1) \in [\![\tau_1]\!] \wedge (p_2, T, v_2) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \,\&\, \tau_2]\!] & \triangleq & \{(p, T, \langle v_1, v_2 \rangle) \mid (p, T, v_1) \in [\![\tau_1]\!] \wedge (p, T, v_2) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \oplus \tau_2]\!] & \triangleq & \{(p, T, \mathsf{inl}(v)) \mid (p, T, v) \in [\![\tau_1]\!]\} \cup \{(p, T, \mathsf{inr}(v)) \mid (p, T, v) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \multimap \tau_2]\!] & \triangleq & \{(p, T, \lambda x.e) \mid \forall p', e', T' {<} T \,.(p', T', e') \in [\![\tau_1]\!]_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in [\![\tau_2]\!]_{\mathcal{E}}\} \\
[\![!_{a<I} \tau]\!] & \triangleq & \{(p, T, !e) \mid \exists p_0, \ldots, p_{I-1} . p_0 + \ldots + p_{I-1} \leqslant p \wedge \forall 0 \leqslant i < I . (p_i, T, e) \in [\![\tau[i/a]]\!]_{\mathcal{E}}\} \\
[\![[n]\, \tau]\!] & \triangleq & \{(p, T, v) \mid \exists p' . p' + n \leqslant p \wedge (p', T, v) \in [\![\tau]\!]\} \\
[\![\mathbb{M}\, n\, \tau]\!] & \triangleq & \{(p, T, v) \mid \forall n', T' {<} T, v' . v \Downarrow_{T'}^{n'} v' \implies \exists p' . n' + p' \leqslant p + n \wedge (p', T - T', v') \in [\![\tau]\!]\} \\
[\![\forall \alpha . \tau]\!] & \triangleq & \{(p, T, \Lambda . e) \mid \forall \tau', T' {<} T \,.(p, T', e) \in [\![\tau[\tau'/\alpha]]\!]_{\mathcal{E}}\} \\
[\![\forall i . \tau]\!] & \triangleq & \{(p, T, \Lambda . e) \mid \forall I \; T' {<} T \,.(p, T', e) \in [\![\tau[I/i]]\!]_{\mathcal{E}}\} \\
[\![c \Rightarrow \tau]\!] & \triangleq & \{(p, T, \Lambda . e) \mid \forall \; T' {<} T \,. \models c \implies (p, T', e) \in [\![\tau]\!]_{\mathcal{E}}\} \\
[\![c \& \tau]\!] & \triangleq & \{(p, T, v) \mid \; \models c \wedge (p, T, v) \in [\![\tau]\!]\} \\
[\![\exists s . \tau]\!] & \triangleq & \{(p, T, v) \mid \exists s' . (p, T, v) \in [\![\tau[s'/s]]\!]\} \\
[\![\lambda_t i . \tau]\!] & \triangleq & f \text{ where } \forall I .\, f \; I = [\![\tau[I/i]]\!] \\
[\![\tau \; I]\!] & \triangleq & [\![\tau]\!] \; I \\
\\
[\![\tau]\!]_{\mathcal{E}} & \triangleq & \{(p, T, e) \mid \forall v, T' {<} T \,. e \Downarrow_{T'} v \implies (p, T - T', v) \in [\![\tau]\!]\}
\end{array}
$$

Definition 7 (Interpretation of typing contexts).

$$
\begin{array}{lcl}
[\![\Gamma]\!]_{\mathcal{E}} & = & \{(p, T, \gamma) \mid \exists f : \mathcal{V}ars \to \mathcal{P}ots . \\
 & & (\forall x \in dom(\Gamma) .\, (f(x), T, \gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}}) \wedge (\sum_{x \in dom(\Gamma)} f(x) \leqslant p)\} \\
[\![\Omega]\!]_{\mathcal{E}} & = & \{(p, T, \delta) \mid \exists f : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots . \\
 & & (\forall (x :_{a<I} \tau) \in \Omega .\, \forall 0 \leqslant i < I .\, (f \; x \; i, T, \delta(x)) \in [\![\tau[i/a]]\!]_{\mathcal{E}}) \wedge \\
 & & (\sum_{x :_{a<I} \tau \in \Omega} \sum_{0 \leqslant i < I} f \; x \; i) \leqslant p\}
\end{array}
$$

Definition 8 (Type and index substitutions). $\sigma : TypeVar \to Type$, $\iota : IndexVar \to Index$

Lemma 9 (Value monotonicity lemma). $\forall p, p', v, \tau, T', T.$
$(p, T, v) \in [\![\tau]\!] \wedge p \leqslant p' \wedge T' {\leqslant} T \implies (p', T', v) \in [\![\tau]\!]$

Proof. Proof by induction on τ □

Lemma 10 (Expression monotonicity lemma). $\forall p, p', v, \tau, T', T.$
$(p, T, e) \in [\![\tau]\!]_{\mathcal{E}} \wedge p \leqslant p' \wedge T' {\leqslant} T \implies (p', T', e) \in [\![\tau]\!]_{\mathcal{E}}$

Proof. From Definition 6 and Lemma 69 □

Lemma 11 (Lemma for substitution). $\forall p, \delta, I, \Omega.$
$(p, \delta) \in [\![\sum_{a<I} \Omega]\!] \implies \exists p_0, \ldots, p_{I-1}.$
$p_0 + \ldots + p_{I-1} \leqslant p \wedge \forall 0 \leqslant i < I . (p_i, \delta) \in [\![\Omega[i/a]]\!]$

Proof. Given: $(p, \delta) \in [\![\sum_{a<I} \Omega]\!]$

When $\Omega = .$

The proof is trivial simply choose p_i as 0 and we are done

When $\Omega(a) = x_0 :_{b<J_0(a)} \tau_0(a), \ldots x_n :_{b<J_n(a)} \tau_n(a)$

Therefore from Definition 2 and Definition 7 we have
$\exists f : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots.$
$(\forall (x_j :_{c<\sum_{a<I} J_j} \sigma) \in (\sum_{a<I} \Omega). \forall 0 \leqslant i < \sum_{a<I} J_j. (f\ x\ i, \delta(x_j)) \in [\![\sigma[i/c]]\!]) \wedge$
$(\sum_{x_j :_{c<\sum_{a<I} J_j} \sigma \in (\sum_{a<I} \Omega)} \sum_{0 \leqslant i < \sum_{a<I} J_j} f\ x_j\ i) \leqslant p \qquad \text{(SM0)}$

To prove the desired, for each $i \in [0, I-1]$ we choose
p_i as $\sum_{x_j :_{b<J_j(i)} \tau_j(i) \in (\Omega(i))} \sum_{0 \leqslant k < J_j(i)} f\ x_j\ (k + \sum_{d<i} J_j[d/i])$
and we need to prove

1. $p_0 + \ldots + p_{I-1} \leqslant p$:
   It suffices to prove that
   $\sum_{0 \leqslant i < I} \sum_{x_j :_{b<J_j(i)} \tau_j(i) \in dom(\Omega(i))} \sum_{0 \leqslant k < J_j(i)} f\ x_j\ (k + \sum_{d<i} J_j(i)[d/i]) \leqslant p$
   We know that $dom(\sum_{a<I} \Omega) = dom(\Omega)$ and from (SM0) we get the desired

2. $\forall 0 \leqslant i < I. (p_i, \delta) \in [\![\Omega[i/a]]\!]$:
   This means given some $0 \leqslant i < I$, from Definition 7 it suffices to prove that
   $\exists f' : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots.$
   $(\forall (x_j :_{b<J_j(i)} \tau_j(i)) \in \Omega[i/a]. \forall 0 \leqslant k < J_j(i). (f'\ x_j\ k, \delta(x_j)) \in [\![\tau_j(i)[k/b]]\!]) \wedge$
   $(\sum_{x_j :_{b<J_j(i)} \in \Omega[i/a]} \sum_{0 \leqslant k < J_j(i)} f'\ x\ k) \leqslant p_i$

   We choose f' s.t
   $\forall x_j :_{b<J_j(i)} \tau_j(i) \in (\Omega[i/a]). \forall 0 \leqslant k < J_j(i). f'\ x_j\ k = f\ x_j\ (k + \sum_{d<i} J_j[d/i])$,

   And we need to prove:

   (a) $\forall (x_j :_{b<J_j(i)} \tau_j(i)) \in \Omega[i/a]. \forall 0 \leqslant k < J_j(i). (f'\ x_j\ k, \delta(x_j)) \in [\![\tau_j(i)[k/b]]\!]$:
   This means given some $(x_j :_{b<J_j(i)} \tau_j(i)) \in \Omega[i/a]$ and some $0 \leqslant k < J_j(i)$ and it suffices to prove that
   $(f'\ x_j\ k, \delta(x_j)) \in [\![\tau_j(i)[k/b]]\!]$

   This means we need to prove that
   $(f\ x_j\ (k + \sum_{d<i} J_j[d/i]), \delta(x_j)) \in [\![\tau_j(i)[k/b]]\!] \qquad \text{(SM1)}$

   Instantiating (SM0) with the given x_j and $(k + \sum_{d<i} J_j[d/i])$ we get
   $(f\ x_j\ (k + \sum_{d<i} J_j[d/i]), \delta(x_j)) \in [\![\sigma[(k + \sum_{d<i} J_j[d/i])/c]]\!]$
   And from Definition 2 we get the desired

   (b) $(\sum_{x_j :_{b<J_j(i)} \tau_j(i) \in \Omega[i/a]} \sum_{0 \leqslant k < J_j(i)} f'\ x\ k) \leqslant p_i$:
   It suffices to prove that
   $(\sum_{x_j :_{b<J_j(i)} \tau_j(i) \in \Omega[i/a]} \sum_{0 \leqslant k < J_j(i)} f\ x\ (k + \sum_{d<i} J_j[d/i])) \leqslant p_i$
   Since we know that p_i is $\sum_{x_j :_{b<J_j(i)} \tau_j(i) \in (\Omega(i))} \sum_{0 \leqslant k < J_j(i)} f\ x_j\ (k + \sum_{d<i} J_j[d/i])$ therefore we are done

□

Theorem 12 (Fundamental theorem)**.** $\forall \Psi, \Theta, \Delta, \Omega, \Gamma, e, \tau \in \mathit{Type}$.
$\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \wedge (p_l, T, \gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}} \wedge (p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}} \wedge . \models \Delta\ \iota \implies$
$(p_l + p_m, T, e\ \gamma\delta) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$.

Proof. Proof by induction on the typing judgment

1. T-var1:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau} \text{ T-var1}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, x : \tau\ \sigma\iota]\!]_{\mathcal{E}}$ and $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l + p_m, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in [\![\Gamma, x : \tau\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Definition 7 we know that $\exists f.(f(x), T, \gamma(x)) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ where $f(x) \leqslant p_l$

Therefore from Lemma 70 we get $(p_l + p_m, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

2. T-var2:

$$\frac{\Theta, \Delta \models I \geqslant 1}{\Psi; \Theta; \Delta; \Omega, x :_{a<I} \tau; \Gamma \vdash x : \tau[0/a]} \text{ T-var2}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$ and $(p_m, T, \delta) \in [\![(\Omega, x :_{a<I} \tau)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l + p_m, x\ \delta\gamma) \in [\![\tau[0/a]\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(p_m, T, \delta) \in [\![(\Omega, x :_{a<I} \tau)\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Definition 7 we know that
$\exists f : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots.$
$((f\ x\ 0, T, \delta(x)) \in [\![\tau[0/a]\ \sigma\iota]\!]_{\mathcal{E}})$ where $(f\ x\ 0) \leqslant p_m$

Therefore from Lemma 70 we get $(p_l + p_m, T, x\ \delta\gamma) \in [\![\tau[0/a]\ \sigma\iota]\!]_{\mathcal{E}}$

3. T-unit:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{ T-unit}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$
To prove: $(p_l + p_m, T, ()\ \delta\gamma) \in [\![\mathbf{1}\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that
$\forall\ T'{<}T\ .()\Downarrow_{T'} () \implies (p_m + p_l, T - T', ()) \in [\![\mathbf{1}]\!]$

This means given $() \Downarrow_0 ()$ it suffices to prove that
$(p_l + p_m, T, ()) \in [\![\mathbf{1}]\!]$

We get this directly from Definition 6

4. T-base:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathsf{b}} \text{ T-base}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, c) \in [\![\mathsf{b}]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v, T'{<}T\ .c \Downarrow_{T'} v \implies (p_m + p_l, T - T', c) \in [\![\mathsf{b}]\!]$

This means given some $v, T'{<}T$ s.t $c \Downarrow_{T'} v$. Also from E-val we know that $T'{=}\ 0$ therefore it suffices to prove that

$(p_l + p_m, T, v) \in [\![\mathsf{b}]\!]$

From (E-val) we know that $v = c$ therefore it suffices to prove that

$(p_l + p_m, T, c) \in [\![\mathsf{b}]\!]$

We get this directly from Definition 6

5. T-nil:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{nil} : L^0\ \tau} \text{ T-nil}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \mathit{nil}\ \delta\gamma) \in [\![L^0\ \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall\ T'{<}T, v'.\mathit{nil} \Downarrow_{T'} v' \implies (p_l + p_m, T - T', v') \in [\![L^0\ \tau\ \sigma\iota]\!]$

This means given some $T'{<}T, v'$ s.t $\mathit{nil} \Downarrow_{T'} v'$ it suffices to prove that

$(p_l + p_m, T - T', v') \in [\![L^0\ \tau\ \sigma\iota]\!]$

From (E-val) we know that $T'{=}\ 0$ and $v' = \mathit{nil}$, therefore it suffices to prove that

$(p_l + p_m, T, \mathit{nil}) \in [\![L^0\ \tau\ \sigma\iota]\!]$

We get this directly from Definition 66

6. T-cons:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : L^n\tau \qquad \Theta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1}\tau} \text{ T-cons}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (e_1 :: e_2)\ \delta\gamma) \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v', t < T\ .(e_1 :: e_2)\ \delta\gamma \Downarrow_t v' \implies (p_l + p_m, T - t, v') \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]$

This means given some $v', t <T$ s.t $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v'$, it suffices to prove that $(p_l + p_m, T - t, v') \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]$

From (E-cons) we know that $\exists v_f, l.v' = v_f :: l$

Therefore from Definition 6 it suffices to prove that

$$\exists p_1, p_2.p_1 + p_2 \leqslant p_l + p_m \wedge (p_1, T - t, v_f) \in [\![\tau\ \sigma\iota]\!] \wedge (p_2, T - t, l) \in [\![L^n \tau\ \sigma\iota]\!] \qquad \text{(F-C0)}$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t $(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that

$\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>:

$(p_{l1} + p_{m1}, T, e_1\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 6 we have

$$\forall t1 < T.e_1\ \delta\gamma \Downarrow v_f \implies (p_{l1} + p_{m1}, T - t1, v_f) \in [\![\tau]\!]$$

Since we are given that $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t1 < t.\ e_1\ \delta\gamma \Downarrow_{t1} v_f$

Therefore we have $(p_{l1} + p_{m1}, T - t1, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-C1)

<u>IH2</u>:

$(p_{l2} + p_{m2}, T, e_2\ \delta\gamma) \in [\![L^n\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 6 we have

$$\forall t2 <T\ .e_2\ \delta\gamma \Downarrow_{t2} l \implies (p_{l2} + p_{m2}, T - t2, l) \in [\![L^n\tau\ \sigma\iota]\!]$$

Since we are given that $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t2 < t - t1.\ e_2\ \delta\gamma \Downarrow l$

Since $t2 < t - t1 < t <T$, therefore we have

$(p_{l2} + p_{m2}, T - t2, l) \in [\![L^n\tau\ \sigma\iota]\!]$ (F-C2)

In order to prove (F-C0) we choose p_1 as $p_{l1} + p_{m1}$ and p_2 as $p_{l2} + p_{m2}$, we get the desired from (F-C1), (F-C2) and Lemma 69

7. T-match:

$$\frac{\begin{array}{c}\Psi;\Theta;\Delta;\Omega_2;\Gamma_1 \vdash e : L^n\ \tau \\ \Psi;\Theta;\Delta, n = 0;\Omega_2;\Gamma_2 \vdash e_1 : \tau' \qquad \Psi;\Theta;\Delta, n > 0;\Omega_2;\Gamma_2, h : \tau, t : L^{n-1}\tau \vdash e_2 : \tau'\end{array}}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2 : \tau'} \text{T-match}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v_f.(\textsf{match } e \textsf{ with } |nil \mapsto e_1 \ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\textsf{match } e \textsf{ with } |nil \mapsto e_1 \ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-M0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that

$\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![L^n\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t' < T\ .e\ \delta\gamma \Downarrow_{t'} v_1 \implies (p_{l1} + p_{m1}, T - t', v_1) \in [\![L^n\tau\ \sigma\iota]\!]$

Since we know that $(\textsf{match } e \textsf{ with } |nil \mapsto e_1 \ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t' < t, v_1.e\ \delta\gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have $(p_{l1} + p_{m1}, T - t', v_1) \in [\![L^n\tau\ \sigma\iota]\!]$

2 cases arise:

(a) $v_1 = nil$:
In this case we know that $n = 0$ therefore
<u>IH2</u>
$(p_{l2} + p_{m2}, T, e_1\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have
$\forall t_1 < T\ .e_1\ \delta\gamma \Downarrow_{t_1} v_f \implies (p_{l2} + p_{m2}, T - t_1, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that $(\textsf{match } e \textsf{ with } |nil \mapsto e_1 \ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_1 < t.\ e_1\ \delta\gamma \Downarrow_{t_1} v_f$.
Since $t_1 < t < T$ therefore we have
$(p_{l2} + p_{m2}, T - t_1, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And from Lemma 69 we get
$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

And finally since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$therefore we get
$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
And we are done

(b) $v_1 = v :: l$:

In this case we know that $n > 0$ therefore

IH2

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T, e_2\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{h \mapsto v\} \cup \{t \mapsto l\}$ and

This means from Definition 6 we have

$\forall t_2 < T .e_2\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that $(\mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_2 < t.\ e_2\ \delta\gamma' \Downarrow v_f$.

Since $t_2 < t < T$ therefore we have

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

From Lemma 69 we get

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And finally since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

And we are done

8. T-existI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \qquad \Theta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S .\tau} \text{ T-existI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v_f.e\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $e\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\exists s.\tau\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\exists s'.(p_l + p_m, T - t, v_f) \in [\![\tau[s'/s]\ \sigma\iota]\!]$ (F-E0)

IH: $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau[n/s]\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t' < T .e\ \delta\gamma \Downarrow_{t'} v_f \implies (p_l + p_m, T - t', v_f) \in [\![\tau[n/s]\ \sigma\iota]\!]$

Since we are given that $e\ \delta\gamma \Downarrow_t v_f$ therefore we get

$(p_l + p_m, T - t, v_f) \in [\![\tau[n/s]\ \sigma\iota]\!]$ (F-E1)

To prove (F-E0) we choose s' as n and we get the desired from (F-E1)

9. T-existsE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : \exists s.\tau \qquad \Psi; \Theta, s; \Delta; \Omega_2; \Gamma_2, x : \tau \vdash e' : \tau' \qquad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{T-existE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (e; x.e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v_f.(e; x.e')\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-EE0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that

$\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 < T\ .e\ \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we know that $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_1 < t, v_1.e\ \delta\gamma \Downarrow v_1$. Therefore we have

$(p_{l1} + p_{m1}, T - t_1, v_1) \in [\![\exists s.\tau\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\exists s'.(p_{l1} + p_{m1}, T - t_1, v_1) \in [\![\tau[s'/s]\ \sigma\iota]\!]$ (F-EE1)

<u>IH2</u>

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T, e'\ \delta'\gamma) \in [\![\tau'\ \sigma\iota']\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto e_1\}$ and $\iota' = \iota \cup \{s \mapsto s'\}$

This means from Definition 6 we have

$\forall t_2 < T\ .e'\ \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota']\!]$

Since we know that $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_2 < t.\ e'\ \delta'\gamma \Downarrow v_f$.

Since $t_2 < t < T$ therefore we have

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in [\![\tau' \ \sigma\iota']\!]$

Since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get
$(p_l + p_m, T - t_2, v_f) \in [\![\tau' \ \sigma\iota']\!]$

And finally from Lemma 69 and since we have $\Psi; \Theta; \Delta \vdash \tau' : K$ therefore we also have
$(p_l + p_m, T - t, v_f) \in [\![\tau' \ \sigma\iota]\!]$
And we are done.

10. T-lam:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x.e : (\tau_1 \multimap \tau_2)} \text{T-lam}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega \ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta \ \iota$
To prove: $(p_l + p_m, T, (\lambda x.e) \ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2) \ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that
$\forall t < T, v_f.(\lambda x.e) \ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![(\tau_1 \multimap \tau_2) \ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\lambda x.e) \ \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\lambda x.e) \ \delta\gamma$. Therefore we have
$(p_l + p_m, T, (\lambda x.e) \ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2) \ \sigma\iota]\!]$

From Definition 6 it suffices to prove that
$\forall p', e', T' < T . (p', T', e') \in [\![\tau_1 \ \sigma\iota]\!]_{\mathcal{E}} \implies (p_l + p_m + p', T', e[e'/x]) \in [\![\tau_2 \ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $p', e', T' < T$ s.t $(p', T', e') \in [\![\tau_1 \ \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that
$(p_l + p_m + p', T', e[e'/x]) \in [\![\tau_2 \ \sigma\iota]\!]_{\mathcal{E}}$ (F-L1)

From IH we know that
$(p_l + p' + p_m, T, e \ \delta\gamma') \in [\![\tau_2 \ \sigma\iota]\!]_{\mathcal{E}}$
where
$\gamma' = \gamma \cup \{x \mapsto e'\}$

Therefore from Lemma 70 we get the desired

11. T-app:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 \ e_2 : \tau_2} \text{T-app}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, \delta) \in [\![(\Omega_1 \oplus \Omega_2) \ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta \ \iota$
To prove: $(p_l + p_m, T, e_1 \ e_2 \ \delta\gamma) \in [\![\tau_2 \ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that
$\forall t < T, v_f.(e_1 \ e_2) \ \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in [\![\tau_2 \ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_m + p_l, T - t, v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-A0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e_1\ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 < T\ .e_1 \Downarrow_{t_1} \lambda x.e \implies (p_{l1} + p_{m1}, T - t_1, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$

Since we know that $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_1 < t.e_1 \Downarrow_{t_1} \lambda x.e$, therefore we have

$(p_{l1} + p_{m1}, T - t_1, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\forall p', e_1, T_1 < T - t_1.(p', T_1, e_1') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p_{l1} + p_{m1} + p', T_1, e[e_1'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-A1)

<u>IH2</u>

$(p_{l2} + p_{m2}, T - t_1 - 1, e_2\ \delta\gamma) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$ (F-A2)

Instantiating (F-A1) with $p_{l2} + p_{m2}$ and $e_2\ \delta\gamma$ we get

$(p_{l1} + p_{m1} + p_{l2} + p_{m2}, T - t_1 - 1, e[e_2\ \delta\gamma/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_2 < T - t_1 - 1.e[e_2\ \delta\gamma/x] \Downarrow_{t_2} v_f \implies (p_l + p_m, T - t_1 - 1 - t_2, v_f) \in [\![\tau_2\ \sigma\iota]\!]$

Since we know that $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_2.e[e_2\ \delta\gamma/x] \Downarrow_{t_2} v_f$, where $t_2 = t - t_1 - 1$, therefore we have

$(p_l + p_m, T - t_1 - t_2 - 1, v_f) \in [\![\tau_2\ \sigma\iota]\!]$

Since from E-app we know that $t = t_1 + t_2 + 1$, this proves (F-A0)

12. T-sub:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{ T-sub}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u> $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH and Lemma 73

13. T-weaken:

$$\frac{\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau \qquad \Psi;\Theta \models \Gamma' <: \Gamma \qquad \Psi;\Theta \models \Omega' <: \Omega}{\Psi;\Theta;\Delta;\Omega';\Gamma' \vdash e : \tau} \text{T-weaken}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma')\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega')\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in [\![(\Gamma')\sigma\iota]\!]_{\mathcal{E}}$ therefore from Lemma 15 we also have $(p_l, T, \gamma) \in [\![(\Gamma)\sigma\iota]\!]_{\mathcal{E}}$

Similarly since we are given that $(p_m, T, \delta) \in [\![(\Omega')\sigma\iota]\!]_{\mathcal{E}}$ therefore from Lemma 16 we also have $(p_m, T, \delta) \in [\![(\Omega)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u>:

$(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH

14. T-tensorI:

$$\frac{\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e_1 : \tau_1 \qquad \Psi;\Theta;\Delta;\Omega_2;\Gamma_2 \vdash e_2 : \tau_1}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash \langle\!\langle e_1, e_2 \rangle\!\rangle : (\tau_1 \otimes \tau_2)} \text{T-tensorI}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega_1 \oplus \Omega_2)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \langle\!\langle e_1, e_2 \rangle\!\rangle\ \delta\gamma) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T\ .\langle\!\langle e_1, e_2 \rangle\!\rangle\ \delta\gamma \Downarrow_t \langle\!\langle v_{f1}, v_{f2} \rangle\!\rangle \implies (p_l + p_m, T - t, \langle\!\langle v_{f1}, v_{f2} \rangle\!\rangle \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$

This means given some $t < T$ s.t $\langle\!\langle e_1, e_2 \rangle\!\rangle\ \delta\gamma \Downarrow_t \langle\!\langle v_{f1}, v_{f2} \rangle\!\rangle$ it suffices to prove that

$(p_l + p_m, T - t, \langle\!\langle v_{f1}, v_{f2} \rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$ (F-TI0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>:

$(p_{l1} + p_{m1}, T, e_1\ \delta\gamma) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 6 we have

$\forall t_1 < T\ .e_1\ \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_{l1} + p_{m1}, T - t_1, v_{f1}) \in [\![\tau_1\ \sigma\iota]\!]$

Since we are given that $\langle\!\langle e_1, e_2 \rangle\!\rangle\ \delta\gamma \Downarrow_t \langle\!\langle v_{f1}, v_{f2} \rangle\!\rangle$ therefore fom E-TI we know that $\exists t_1 < t.e_1\ \delta\gamma \Downarrow_{t_1} v_{f1}$

Hence we have $(p_{l1} + p_{m1}, T - t_1, v_{f1}) \in [\![\tau_1\ \sigma\iota]\!]$ (F-TI1)

<u>IH2</u>:

$(p_{l2} + p_{m2}, T, e_2\ \delta\gamma) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 6 we have

$\forall t_2 < T\ .e_2\ \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_{l2} + p_{m2}, T - t_2, v_{f2} \in [\![\tau_2\ \sigma\iota]\!]$

Since we are given that $\langle\!\langle e_1, e_2\rangle\!\rangle\ \delta\gamma \Downarrow_t \langle\!\langle v_{f1}, v_{f2}\rangle\!\rangle$ therefore fom E-TI we also know that $\exists t_2 < t.e_2\ \delta\gamma \Downarrow_{t_2} v_{f2}$

Since $t_2 < t < T$ therefore we have

$(p_{l2} + p_{m2}, T - t_2, v_{f2}) \in [\![\tau_2\ \sigma\iota]\!]$ (F-TI2)

Applying Lemma 69 on (F-TI1) and (F-TI2) and by using Definition 66 we get the desired.

15. T-tensorE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \qquad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e \mathsf{\ in\ } e' : \tau} \text{ T-tensorE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-TE0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .e\ \delta\gamma \Downarrow_{t_1} \langle\!\langle v_1, v_2\rangle\!\rangle\ \delta\gamma \implies (p_{l1} + p_{m1}, T - t_1, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_1 < t, v_1, v_2.e\ \delta\gamma \Downarrow_{t_1} \langle\!\langle v_1, v_2\rangle\!\rangle$. Therefore we have

$(p_{l1} + p_{m1}, T - t_1, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 we know that

$\exists p_1, p_2.p_1 + p_2 \leqslant p_{l1} + p_{m1} \wedge (p_1, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p_2, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-TE1)

IH2

$(p_{l2} + p_{m2} + p_1 + p_2, T, e'\ \delta\gamma') \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v_1\} \cup \{y \mapsto v_2\}$

This means from Definition 66 we have

$\forall t_2 < T\ .e'\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_1 + p_2, T - t_2, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e\ \mathsf{in}\ e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_2 < t.e'\ \delta\gamma' \Downarrow_{t_2} v_f$. Therefore we have

$(p_{l2} + p_{m2} + p_1 + p_2, T - t_2, v_f) \in [\![\tau\ \sigma\iota]\!]$

From Lemma 69 we get

$(p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

And we are done

16. T-withI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2\rangle : (\tau_1 \ \&\ \tau_2)}\ \text{T-withI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \langle e_1, e_2\rangle\ \delta\gamma) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T\ .\langle e_1, e_2\rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2}\rangle \implies (p_l + p_m, T - t, \langle v_{f1}, v_{f2}\rangle \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]$

This means given $\langle e_1, e_2\rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2}\rangle$ it suffices to prove that

$(p_l + p_m, T - t, \langle v_{f1}, v_{f2}\rangle) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]$ (F-WI0)

IH1:

$(p_l + p_m, T, e_1\ \delta\gamma) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_1 < T\ .e_1\ \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_l + p_m, T - t_1, v_{f1}) \in [\![\tau_1\ \sigma\iota]\!]$

Since we are given that $\langle e_1, e_2\rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2}\rangle$ therefore fom E-WI we know that $\exists t_1 < t.e_1\ \delta\gamma \Downarrow_{t_1} v_{f1}$

Since $t_1 < t < T$, therefore we have

$(p_l + p_m, T - t_1, v_{f1}) \in [\![\tau_1\ \sigma\iota]\!]$ (F-WI1)

IH2:

$(p_l + p_m, T, e_2\ \delta\gamma) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_2 < T . e_2 \ \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_l + p_m, T - t_2, v_{f2} \in [\![\tau_2 \ \sigma\iota]\!]$

Since we are given that $\langle e_1, e_2 \rangle \ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ therefore fom E-WI we also know that $\exists t_2 < t . e_2 \ \delta\gamma \Downarrow_{t_2} v_{f2}$

Since $t_2 < t < T$, therefore we have

$(p_l + p_m, T - t_2, v_{f2}) \in [\![\tau_2 \ \sigma\iota]\!]$ (F-WI2)

Applying Lemma 69 on (F-W1) and (F-W2) we get the desired.

17. T-fst:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \ \& \ \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{fst}(e) : \tau_1} \text{ T-fst}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma) \ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega \ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\mathsf{fst}(e)) \ \delta\gamma) \in [\![\tau_1 \ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f . (\mathsf{fst}(e)) \ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![\tau_1 \ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{fst}(e)) \ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau_1 \ \sigma\iota]\!]$ (F-F0)

<u>IH</u>

$(p_l + p_m, T, e \ \delta\gamma) \in [\![(\tau_1 \ \& \ \tau_2) \ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T . e \ \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle \ \delta\gamma \implies (p_l + p_m, T - t_1, \langle v_1, v_2 \rangle) \in [\![(\tau_1 \ \& \ \tau_2) \ \sigma\iota]\!]$

Since we know that $(\mathsf{fst}(e)) \ \delta\gamma \Downarrow_t v_f$ therefore from E-fst we know that $\exists t_1 < t . v_1, v_2 . e \ \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle$.

Since $t_1 < t < T$, therefore we have

$(p_l + p_m, T - t_1, \langle v_1, v_2 \rangle) \in [\![(\tau_1 \ \& \ \tau_2) \ \sigma\iota]\!]$

From Definition 66 we know that

$(p_l + p_m, T - t_1, v_1) \in [\![\tau_1 \ \sigma\iota]\!]$

Finally using Lemma 69 we also have

$(p_l + p_m, T - t, v_1) \in [\![\tau_1 \ \sigma\iota]\!]$

Since from E-fst we know that $v_f = v_1$, therefore we are done.

18. T-snd:

Similar reasoning as in T-fst case above.

19. T-inl:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{inl}(e) : \tau_1 \oplus \tau_2} \text{ T-inl}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega\ \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \mathsf{inl}(e)\ \delta\gamma) \in \llbracket (\tau_1 \oplus \tau_2)\ \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T\ .\mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v) \implies (p_l + p_m, T - t, \mathsf{inl}(v) \in \llbracket (\tau_1 \oplus \tau_2)\ \sigma\iota \rrbracket$

This means given some $t < T$ s.t $\mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v)$ it suffices to prove that

$(p_l + p_m, T - t, \mathsf{inl}(v)) \in \llbracket (\tau_1 \oplus \tau_2)\ \sigma\iota \rrbracket$ (F-IL0)

<u>IH</u>:

$(p_l + p_m, T, e_1\ \delta\gamma) \in \llbracket \tau_1\ \sigma\iota \rrbracket_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_1 < T\ .e_1\ \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_l + p_m, T - t_1, v_{f1}) \in \llbracket \tau_1\ \sigma\iota \rrbracket$

Since we are given that $\mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v)$ therefore fom E-inl we know that $\exists t_1 < t.e\ \delta\gamma \Downarrow_{t_1} v$

Hence we have $(p_l + p_m, T - t_1, v) \in \llbracket \tau_1\ \sigma\iota \rrbracket$

From Lemma 69 we get $(p_l + p_m, T - t, v) \in \llbracket \tau_1\ \sigma\iota \rrbracket$

And finally from Definition 66 we get (F-IL0)

20. T-inr:

Similar reasoning as in T-inr case above.

21. T-case:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1; e_2 : \tau} \text{ T-case}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2)\ \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega\ \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma) \in \llbracket \tau\ \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau\ \sigma\iota \rrbracket$

This means given some $t < T, v_f$ s.t $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in \llbracket \tau\ \sigma\iota \rrbracket$ (F-C0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1)\sigma\iota \rrbracket_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in \llbracket (\Gamma_2)\sigma\iota \rrbracket_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t' < T\ .e\ \delta\gamma \Downarrow_{t'} v_1\ \delta\gamma \implies (p_{l1} + p_{m1}, T - t', v_1) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$

Since we know that $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t' < t, v_1.e\ \delta\gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have

$(p_{l1} + p_{m1}, T - t', v_1) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$

2 cases arise:

(a) $v_1 = \mathsf{inl}(v)$:

<u>IH2</u>

$(p_{l2} + p_{m2}p_{l1} + p_{m1}, T - t', e_1\ \delta\gamma') \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v\}$

This means from Definition 66 we have

$\forall t_1 < T - t'.e_1\ \delta\gamma' \Downarrow_{t_1} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t' - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t_1.e_1\ \delta\gamma' \Downarrow v_f$ where $t_1 = t - t' - 1$.

Since $t_1 = t - t' - 1 < T - t'$ therefore we have

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t' - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

From Lemma 69 we get

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

And we are done

(b) $v_1 = \mathsf{inr}(v)$:

Similar reasoning as in the inl case above.

22. T-subExpI:

$$\frac{\Psi; \Theta, a; \Delta, a < I; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \sum_{a<I} \Omega; . \vdash !e : !_{a<I}\tau}\ \text{T-subExpI}$$

Given: $(p_l, \gamma) \in [\![.]\!]_{\mathcal{E}}$, $(p_m, \delta) \in [\![(\sum_{a<I} \Omega)\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, !e\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T\ .(!e)\ \delta\gamma \Downarrow_t (!e)\ \delta\gamma \implies (p_m + p_l, T - t, (!e)\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]$

This means given some $t < T$. s.t $(!e)\ \delta\gamma \Downarrow_t (!e)\ \delta\gamma$ it suffices to prove that
$(p_m + p_l, T - t, (!e)\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that
$\exists p_0, \ldots, p_{I-1}.p_0 + \ldots + p_{I-1} \leqslant (p_m + p_l) \wedge \forall 0 \leqslant i < I.(p_i, T, e\ \delta\gamma) \in [\![\tau[i/a]]\!]_{\mathcal{E}}$ (F-SI0)

Since we know that $(p_m, T, \delta) \in [\![(\sum_{a<I}\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Lemma 11 we know that
$\exists p'_0, \ldots, p'_{I-1}.\ p'_0 + \ldots + p'_{I-1} \leqslant p_m \wedge \forall 0 \leqslant i < I.(p_i, T, \delta) \in [\![\Omega[i/a]]\!]_{\mathcal{E}}$ (F-SI1)

Instantiating IH with each $p'_0 \ldots p'_{I-1}$ we get
$(p'_0, T, e\ \delta\gamma) \in [\![\tau[0/a]\ \sigma\iota]\!]_{\mathcal{E}}$ and
...
$(p'_{I-1}, T, e\ \delta\gamma) \in [\![\tau[I-1/a]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SI2)

Therefore we get (F-SI0) from (F-SI1) and (F-SI2)

23. T-subExpE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (!_{a<I}\tau) \qquad \Psi; \Theta; \Delta; \Omega_2, x :_{a<I} \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\,!\,x = e \mathsf{\ in\ } e' : \tau'} \text{ T-subExpE}$$

Given: $(p_l, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, \delta) \in [\![(\Omega_1 \oplus \Omega_2)\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$
To prove: $(p_l + p_m, (\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that
$\forall t < T, v_f.(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t < T$ s.t. $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_m + p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-SE0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t
$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t
$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>
$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have
$\forall t_1 < T . e\ \delta\gamma \Downarrow_{t_1} !e_1\ \delta\gamma \implies (p_{l1} + p_{m1}, T - t_1, !e_1\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]$

Sice we know that $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_1 < t, e_1.e\ \delta\gamma \Downarrow_{t_1} !e_1\ \delta\gamma$. Therefore we have
$(p_{l1} + p_{m1}, T - t_1, !e_1\ \delta\gamma) \in [\![!_{a<I}\tau\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\exists p_0, \ldots, p_{I-1}.p_0 + \ldots + p_{I-1} \leqslant (p_{l1} + p_{m1}) \wedge \forall 0 \leqslant i < I.(p_i, T - t_1, e_1\ \delta\gamma) \in [\![\tau[i/a]]\!]_{\mathcal{E}}$ (F-SE1)

<u>IH2</u>

$(p_{l2} + p_{m2} + p_0 + \ldots + p_{I-1}, T - t_1, e'\ \delta'\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto e_1\}$

This means from Definition 6 we have

$\forall t_2 < T - t_1.e'\ \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_0 + \ldots + p_{I-1}, T - t_1 - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_2.e'\ \delta'\gamma \Downarrow v_f$ s.t. $t_2 = t - t_1 - 1$. Therefore we have

$(p_{l2} + p_{m2} + p_0 + \ldots + p_{I-1}, T - t_1 - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since from (F-SE1) we know that $p_0 + \ldots + p_{I-1} \leqslant p_{l1} + p_{m1}$ therefore from Lemma 70 we get

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And finally since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And we are done

24. T-tabs:

$$\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K\ .\tau)} \text{ T-tabs}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(\forall \alpha : K\ .\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v.\Lambda.e\ \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in [\![(\forall \alpha : K\ .\tau)\ \sigma\iota]\!]$

This means given some v s.t $\Lambda.e\ \delta\gamma \Downarrow v$ and from (E-val) we know that $v = \Lambda.e\ \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(\forall \alpha : K\ .\tau)\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\forall \tau', T' < T\ .(p_l + p_m, T', e\ \delta\gamma) \in [\![\tau[\tau'/\alpha]\sigma\iota]\!]_{\mathcal{E}}$

This means given some $\tau', T' < T$ it suffices to prove that

$(p_l + p_m, T', e\ \delta\gamma) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-TAb0)

<u>IH</u> $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma'\iota]\!]_{\mathcal{E}}$

where

$\sigma' = \sigma \cup \{\alpha \mapsto \tau'\}$

We get the desired directly from IH

25. T-tapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha : K . \tau) \qquad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\ [] : (\tau[\tau'/\alpha])} \text{ T-tapp}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma \sigma \iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma \iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, e\ []\ \delta\gamma) \in [\![(\tau[\tau'/\alpha])\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v_f.(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in [\![(\tau[\tau'/\alpha])\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_m + p_l, T - t, v_f) \in [\![(\tau[\tau'/\alpha])\ \sigma\iota]\!]$ (F-Tap0)

<u>IH</u>

$(p_l + p_m, T, e\ \delta\gamma) \in [\![(\forall \alpha.\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 < T, v'.e\ \delta\gamma \Downarrow_{t_1} v' \implies (p_l + p_m, T - t_1, v') \in [\![(\forall \alpha \tau)\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t.e\delta\gamma \Downarrow_{t_1} \Lambda.e$, therefore we have

$(p_l + p_m, T - t_1, \Lambda.e) \in [\![(\forall \alpha.\tau)\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\forall \tau'', T_1 < T - t_1.(p_l + p_m, T - t_1 - T_1, e\ \delta\gamma) \in [\![\tau[\tau''/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$

Instantiating it with the given τ' and $T - t_1 - 1$ we get

$(p_l + p_m, T - t_1 - 1, e\ \delta\gamma) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 we know that

$\forall t_2 < T - t_1 - 1, v''.e\ \delta\gamma \Downarrow_{t_2} v'' \implies (p_l + p_m, T - t_1 - 1 - t_2, v'') \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_2.e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$(p_l + p_m, T - t, v_f) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$

And we are done.

26. T-iabs:

$$\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S . \tau)} \text{ T-iabs}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(\forall i : S . \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v.\Lambda.e\ \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in [\![(\forall i : S . \tau)\ \sigma\iota]\!]$

This means given some $t < T, v$ s.t $\Lambda.e\ \delta\gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e\ \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(\forall i : S . \tau)\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\forall I.(p_l + p_m, T, e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some I it suffices to prove that

$(p_l + p_m, T, e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-TAb0)

<u>IH</u> $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota']\!]_{\mathcal{E}}$

where

$\iota' = \iota \cup \{i \mapsto I\}$

We get the desired directly from IH

27. T-iapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S . \tau) \qquad \Theta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\ [] : (\tau[I\ /i])} \text{ T-iapp}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, e\ []\ \delta\gamma) \in [\![(\tau[I\ /i])\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v_f.(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in [\![(\tau[I\ /i])\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_m + p_l, T - t, v_f) \in [\![(\tau[I\ /i])\ \sigma\iota]\!]$ (F-Iap0)

<u>IH</u>

$(p_l + p_m, T, e\ \delta\gamma) \in [\![(\forall i : S . \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 < T, v'.e\ \delta\gamma \Downarrow_{t_1} v' \implies (p_l + p_m, T - t_1, v') \in [\![(\forall i : S . \tau)\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from (E-iapp) we know that $\exists t_1 < t.e\delta\gamma \Downarrow_{t_1} \Lambda.e$, therefore we have

$(p_l + p_m, T - t_1, \Lambda.e) \in [\![(\forall i : S\ .\tau)\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\forall I'', T_1 < T - t_1.(p_l + p_m, T - t_1 - T_1, e\ \delta\gamma) \in [\![\tau[I''/i]\ \sigma\iota]\!]_{\mathcal{E}}$

Instantiating it with the given I and $T - t_1 - 1$ we get

$(p_l + p_m, T - t_1 - 1, e\ \delta\gamma) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 we know that

$\forall v'', t_2 < T - t_1 - 1.e\ \delta\gamma \Downarrow_{t_2} v'' \implies (p_l + p_m, T - t_1 - 1 - t_2, v'') \in [\![\tau[I/i]\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-iapp we know that $\exists t_2.e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$(p_l + p_m, v_f) \in [\![\tau[I/i]\ \sigma\iota]\!]$

And we are done.

28. T-CI:

$$\frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (c \Rightarrow \tau)} \text{ T-CI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v, t < T\ .\Lambda.e\ \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

This means given some $v, t < T$ s.t $\Lambda.e\ \delta\gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e\ \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$(p_l + p_m, T, \Lambda.e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\forall\ T' < T\ . \models c\ \iota \implies (p_l + p_m, T', e\ \delta\gamma) \in [\![\tau\sigma\iota]\!]_{\mathcal{E}}$

This means given some $T' < T$ s.t. . $\models c\ \iota$ it suffices to prove that

$(p_l + p_m, T', e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u> $(p_l + p_m, T', e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH

29. T-CE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e[] : \tau} \text{ T-CE}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma \sigma \iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, e\ []\ \delta\gamma) \in [\![(\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v_f, t < T\ .(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in [\![(\tau)\ \sigma\iota]\!]$

This means given some $v_f, t < T$ s.t. $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_m + p_l, T - t, v_f) \in [\![(\tau)\ \sigma\iota]\!]$ (F-Tap0)

IH

$(p_l + p_m, T, e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall v', t' < T\ .e\ \delta\gamma \Downarrow_{t'} v' \implies (p_l + p_m, T - t', v') \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-CE we know that $\exists t' < t.e\delta\gamma \Downarrow_{t'} \Lambda.e'$, therefore we have

$(p_l + p_m, T - t', \Lambda.e') \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

Therefore from Definition 6 we have

$\forall t'' < T - t'. \models c\ \iota \implies (p_l + p_m, T - t' - t'', e'\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given $\Theta; \Delta \models c$ and $. \models \Delta\ \iota$. Therefore instantiating it with $T - t' - 1$ and since we know that $. \models c\ \iota$. Hence we get

$(p_l + p_m, T - t' - 1, e'\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall v'_f, t'' < T - t' - 1.(e')\ \delta\gamma \Downarrow v'_f \implies (p_m + p_l, v'_f) \in [\![(\tau)\ \sigma\iota]\!]$

Since from E-CE we know that $e'\delta\gamma \Downarrow_t v_f$ therefore we know that $\exists t''.e'\ \delta\gamma \Downarrow_{t''} v_f$ s.t $t = t' + t'' + 1$

Therefore instantiating (F-CE1) with the given v_f and t'' we get

$(p_m + p_l, T - t, v_f) \in [\![(\tau)\ \sigma\iota]\!]$

and we are done.

30. T-CAndI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{ T-CAndI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e\ \delta\gamma) \in [\![c \& \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v_f, t <T .e\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f\ \delta\gamma) \in [\![c\&\tau\ \sigma\iota]\!]$

This means given some $v_f, t <T$ s.t $e\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![c\&\tau\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$. \models c\iota \wedge (p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we are given that $. \models \Delta\iota$ and $\Theta; \Delta \models c$ therefore it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-CAI0)

<u>IH</u>: $(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t' <T .e\ \delta\gamma \Downarrow_{t'} v_f \implies (p_l + p_m, T - t', v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we are given that $e\ \delta\gamma \Downarrow_t v_f$ therefore we get

$(p_l + p_m, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-CAI1)

We get the desired from (F-CAI1)

31. T-CAndE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (c\&\tau) \qquad \Psi; \Theta; \Delta, c; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{clet}\, x = e\ \mathsf{in}\ e' : \tau'} \text{ T-CAndE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\mathsf{clet}\, x = e\ \mathsf{in}\ e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall v_f, t <T .(\mathsf{clet}\, x = e\ \mathsf{in}\ e')\ \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $v_f, t <T$ s.t. $(\mathsf{clet}\, x = e\ \mathsf{in}\ e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-CAE0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that

$\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e\ \delta\gamma) \in [\![c\&\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 <T .e\ \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1 v_1) \in [\![c\&\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we know that $(\mathsf{clet}\, x = e \;\mathsf{in}\; e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_1 < t, v_1.e\ \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have

$(p_{l1} + p_{m1}, T - t_1, v_1) \in \llbracket c\&\tau\ \sigma\iota \rrbracket$

Therefore from Definition 6 we have

$. \models c\iota \wedge (p_{l1} + p_{m1}, T - t_1, v_1) \in \llbracket \tau\ \sigma\iota \rrbracket$ (F-CAE1)

<u>IH2</u>

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_1, e'\ \delta\gamma') \in \llbracket \tau'\ \sigma\iota \rrbracket_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v_1\}$

This means from Definition 6 we have

$\forall t_2 < T\ .e'\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_1 - t_2, v_f) \in \llbracket \tau'\ \sigma\iota \rrbracket$

Since we know that $(\mathsf{clet}\, x = e \;\mathsf{in}\; e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_2.e'\ \delta'\gamma \Downarrow_{t_2} v_f$ s.t $t_2 = t - t_1 - 1$

Therefore we have

$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_1 - t_2, v_f) \in \llbracket \tau'\ \sigma\iota' \rrbracket$

Since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$(p_l + p_m, T - t, v_f) \in \llbracket \tau'\ \sigma\iota' \rrbracket$

And we are done.

32. T-fix:

$$\frac{\Psi; \Theta, b; \Delta, b < L; \Omega, x :_{a<I} \tau[(b + 1 + \bigtriangleup_b^{b+1,a} I)/b]; . \vdash e : \tau \qquad L \geqslant \bigtriangleup_b^{0,1} I}{\Psi; \Theta; \Delta; \sum_{b<L} \Omega; . \vdash \mathsf{fix} x.e : \tau[0/b]} \text{ T-fix}$$

Given: $(p_l, T, \gamma) \in \llbracket . \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \sum_{b<L} \Omega\ \sigma\iota \rrbracket_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, (\mathsf{fix} x.e)\ \delta\gamma) \in \llbracket \tau[0/b]\ \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall\, T' < T, v_f.(\mathsf{fix} x.e)\ \delta\gamma \Downarrow_{T'} v_f \implies (p_m + p_l, T - T', v_f) \in \llbracket \tau[0/b]\ \sigma\iota \rrbracket$

This means given some $t < T, v_f$ s.t. $\mathsf{fix} x.e\ \delta\gamma \Downarrow_{T'} v_f$ therefore it suffices to prove that

$(p_l + p_m, T - T', v_f) \in \llbracket \tau[0/b]\ \sigma\iota \rrbracket$ (F-FX0)

Also from Lemma 11 we know that

$\exists p'_0, \ldots, p'_{(I-1)}.\ p'_0 + \ldots + p'_{(L-1)} \leqslant p_m \wedge \forall 0 \leqslant i < L.(p_i, \delta) \in \llbracket \Omega[i/a] \rrbracket_{\mathcal{E}}$

We define

$$
\begin{array}{lcl}
p_N(leaf) & \triangleq & p'_{leaf} \\
p_N(t) & \triangleq & p'_t + (\sum_{a<I(t)} p_N((t+1+\bigcirc\!\!\!\!\triangle_{b}^{t+1,a} I(b))))
\end{array}
$$

<u>Claim</u>

$\forall 0 \leqslant t < L.\ (p_N(t), T, e\ \delta'\gamma) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto \mathsf{fix}x.e\delta\}$

This means given some t it suffices to prove

$(p_N(t), T, e\ \delta'\gamma) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

We prove this by induction on t

<u>Base case</u>: when t is a leaf node (say l)

It suffices to prove that $(p'_l, T, e\ \delta'\gamma) \in [\![\tau[l/b]\ \sigma\iota]\!]_{\mathcal{E}}$

We know that $I(l) = 0$ therefore from IH (of the outer induction) we get the desired

<u>Inductive case</u>: when t is some arbitrary non-leaf node

From IH we know that

$\forall a < I(t).(p_N(t'), T, e\ \delta'\gamma) \in [\![\tau[t'/b]\ \sigma\iota]\!]_{\mathcal{E}}$ where $t' = (t+1+\bigcirc\!\!\!\!\triangle_{b}^{t+1,a} I(b))$

<u>Claim</u>

$\forall \tau'.(p_N(t'), T, e\ \delta'\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$ where $\delta' = \delta \cup \{x \mapsto \mathsf{fix}x.e\delta\} \implies$

$(p_N(t'), T, \mathsf{fix}x.e\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

Proof is trivial

$\square$

Therefore we have

$\forall a < I(t).(p_N(t'), T, \mathsf{fix}x.e\ \delta\gamma) \in [\![\tau[t'/b]\ \sigma\iota]\!]_{\mathcal{E}}$ where $t' = (t+1+\bigcirc\!\!\!\!\triangle_{b}^{t+1,a} I(b))$

Now from the IH of the outer induction we get

$(p'_t + \sum_{a<I} p_N(t'), T, e\ \delta'\gamma) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

Which means we get the desired i.e

$(p_N(t), T, e\ \delta'\gamma) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

$\square$

Since we have proved

$\forall 0 \leqslant t < L.\ (p_N(t), T, e\ \delta'\gamma) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto \mathsf{fix} x.e\}$

Therefore from Definition 6 we have

$\forall 0 \leqslant t < L.\ \forall\, T''{<}T\ .e\ \delta'\gamma \Downarrow_{T''} v_f \implies (p_N(t), T - T'', v_f) \in [\![\tau[t/b]\ \sigma\iota]\!]_{\mathcal{E}}$

Instantiating with t with 0 and since we know that $\mathsf{fix} x.e\ \delta\gamma \Downarrow_{T'} v_f$ therefore knwo that $\exists\, T''{<}T'\ .e\ \delta'\gamma \Downarrow_{T''} v_f$ where $T''{=}T'\ {-}1$

$(p_N(0), T - T'', v_f) \in [\![\tau[0/b]\ \sigma\iota]\!]_{\mathcal{E}}$

Since $p_N(0) \leqslant p_m$ therefore $p_N(0) \leqslant p_l + p_m$

And we get the (F-FX0) from Lemma 69

33. T-ret:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{ret}\, e : \mathbb{M}\, 0\, \tau} \text{T-ret}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \mathsf{ret}\, e\ \delta\gamma) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t\ {<}T\ .(\mathsf{ret}\, e)\ \delta\gamma \Downarrow (\mathsf{ret}\, e)\ \delta\gamma \implies (p_m + p_l, T\ {-}t, (\mathsf{ret}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]$

Since from E-val we know that $t = 0$ therefore it suffices to prove that

$(p_m + p_l, T, (\mathsf{ret}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\forall n', t'\ {<}T, v_f.(\mathsf{ret}\, e)\ \delta\gamma \Downarrow^{n'}_{t'} v_f \implies \exists p'.n' + p' \leqslant p_l + p_m \wedge (p', T\ {-}t', v_f) \in [\![\tau]\!]$

This means given some $n', t'\ {<}T, v_f$ s.t. $(\mathsf{ret}\, e)\ \delta\gamma \Downarrow^{n'}_{t'} v_f$ it suffices to prove that

$\exists p'.n' + p' \leqslant p_l + p_m \wedge (p', T\ {-}t', v_f) \in [\![\tau]\!]$

From (E-ret) we know that $n' = 0$ therefore we choose p' as $p_l + p_m$ and it suffices to prove that

$(p_l + p_m, T\ {-}t', v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-R0)

<u>IH</u>

$(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1\ {<}T\ .(e)\ \delta\gamma \Downarrow_{t_1} v_f \implies (p_m + p_l, T\ {-}t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{ret}\, e)\ \delta\gamma \Downarrow^{0}_{t'} v_f$ therefore from (E-ret) we know that $\exists t_1 < t.e\ \delta\gamma \Downarrow_{t''} v_f$ s.t $t_1 + 1 = t'$

Therefore we have $(p_m + p_l, T\ {-}t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$ and from Lemma 69 we are done

34. T-bind:

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \mathbb{M}\, n_1\, \tau_1 \\ \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}\, n_2\, \tau_2 \qquad \Theta \vdash n_1 : \mathbb{R}^+ \qquad \Theta \vdash n_2 : \mathbb{R}^+\end{array}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2 : \mathbb{M}(n_1 + n_2)\, \tau_2} \text{ T-bind}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega_1 \oplus \Omega_2)\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2\ \delta\gamma) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$\forall t < T, v.(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, (\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

This means given some $t < T, v$ s.t. $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow_t v$ and from E-val we know that $v = (\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma$ and $t = 0$. It suffices to prove that

$(p_m + p_l, T, (\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 6 it suffices to prove that

$\forall s', t' < T, v_f.(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2\ \delta\gamma) \Downarrow_{t'}^{s'} v_f \implies \exists p'.s' + p' \leqslant p_l + p_m + n \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$

This means given some $s', t' < T, v_f$ s.t $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2\ \delta\gamma) \Downarrow_{t'}^{s'} v_f$ and we need to prove that

$\exists p'.s' + p' \leqslant p_l + p_m + n \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-B0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e_1\ \delta\gamma) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it means we have

$\forall t_1 < T\ .(e_1)\ \delta\gamma \Downarrow_{t_1} (e_1)\ \delta\gamma \implies (p_{m1} + p_{l1}, T - t_1, (e_1)\ \delta\gamma) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-bind we know that $\exists t_1 < t', v_{m1}.(e_1)\ \delta\gamma \Downarrow (e_1)\ \delta\gamma$.

Since $t_1 < t' < T$, therefore we have

$(p_{m1} + p_{l1}, T - t_1, (e_1)\ \delta\gamma) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]$

This means from Definition 6 we are given that

$\forall t_1' < T - t_1.(e_1\ \delta\gamma) \Downarrow_{t_1'}^{s_1} v_1 \implies \exists p_1'.s_1 + p_1' \leqslant p_{l1} + p_{m1} + n_1 \wedge (p_1', T - t_1 - t_1', v_1) \in [\![\tau_1\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow_{t'} v_f$ therefore from E-bind we know that $\exists t_1' < t' - t_1.(e_1)\ \delta\gamma \Downarrow_{t_1'}^{s_1} v_1$.

This means we have

$\exists p'_1.s_1 + p'_1 \leqslant p_{l1} + p_{m1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in [\![\tau_1\ \sigma\iota]\!]$ (F-B1)

<u>IH2</u>

$(p_{l2} + p_{m2} + p'_1, T - t_1 - t'_1, e_2\ \delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_2)\,\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it means we have

$\forall t_2 < T - t_1 - t'_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \implies (p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t'_1 - t_2, (e_2)\ \delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_2)\,\tau_2\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow - \Downarrow^{-}_{t} v_f$ therefore from E-bind we know that $\exists t_2 < t' - t_1 - t'_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2)\ \delta\gamma \cup \{x \mapsto v_1\}$.

Since $t_2 < t' - t_1 - t'_1 < T - t_1 - t'_1$ therefore we have

$(p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t'_1 - t_2, (e_2)\ \delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_2)\,\tau_2\ \sigma\iota]\!]$

This means from Definition 6 we are given that

$\forall t'_2 < T - t_1 - t'_1 - t_2.(e_2\ \delta\gamma \cup \{x \mapsto v_1\}) \Downarrow^{s_2}_{t'_2} v_2 \implies \exists p'_2.s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow - \Downarrow^{-}_{t'} v_f$ therefore from E-bind we know that $\exists t'_2 < t' - t_1 - t'_1 - t_2, s_2, v_2.v_{m2} \Downarrow^{s_2}_{t'_2} v_2$.

This means we have

$\exists p'_2.s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-B2)

In order to prove (F-B0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leqslant p_l + p_m + n$:
Since from (F-B2) we know that
$s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p'_1 + n_2$

Adding s_1 on both sides we get
$s_1 + s_2 + p'_2 \leqslant p_{l2} + p_{m2} + s_1 + p'_1 + n_2$

Since from (F-B1) we know that
$s_1 + p'_1 \leqslant p_{l1} + p_{m1} + n_1$

therefore we also have
$s_1 + s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p_{l1} + p_{m1} + n_1 + n_2$

And finally since we know that $n = n_1 + n_2$, $s' = s_1 + s_2$, $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get the desired

(b) $(p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_f) \in [\![\tau_2\ \sigma\iota]\!]$:
From E-bind we know that $v_f = v_2$ therefore we get the desired from (F-B2)

35. T-tick:

$$\frac{\Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^n : \mathbb{M}\, n\, \mathbf{1}} \text{ T-tick}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma \sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \uparrow^n\ \delta\gamma) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$(\uparrow^n)\ \delta\gamma \Downarrow_0 (\uparrow^n)\ \delta\gamma \implies (p_m + p_l, T, (\uparrow^n)\ \delta\gamma) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]$

It suffices to prove that

$(p_m + p_l, T, (\uparrow^n)\ \delta\gamma) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\forall t' < T, n'.(\uparrow^n)\ \delta\gamma \Downarrow_{t'}^{n'} () \implies \exists p'.n' + p' \leqslant p_l + p_m + n \wedge (p', T - t', ()) \in [\![\mathbf{1}]\!]$

This means given some $t' < T, n'$ s.t. $(\uparrow^n)\ \delta\gamma \Downarrow_{t'}^{n'} ()$ it suffices to prove that

$\exists p'.n' + p' \leqslant p_l + p_m + n \wedge (p', T - t', ()) \in [\![\mathbf{1}]\!]$

From (E-tick) we know that $n' = n$ therefore we choose p' as $p_l + p_m$ and it suffices to prove that

$(p_l + p_m, T - t', ()) \in [\![\mathbf{1}]\!]$

We get this directly from Definition 6

36. T-release:

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : [n_1]\, \tau_1 \\ \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(n_1 + n_2)\, \tau_2 \qquad \Theta \vdash n_1 : \mathbb{R}^+ \qquad \Theta \vdash n_2 : \mathbb{R}^+\end{array}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2 : \mathbb{M}\, n_2\, \tau_2} \text{ T-release}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![(\Omega_1 \oplus \Omega_2)\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l + p_m, T, \mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2\ \delta\gamma) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$(\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2)\ \delta\gamma \Downarrow_0 (\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2\ \delta\gamma) \implies (p_m + p_l, (\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2)\ \delta\gamma) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

This means given $(\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2)\ \delta\gamma \Downarrow_0 (\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2)\ \delta\gamma$ it suffices to prove that

$(p_m + p_l, (\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2)\ \delta\gamma) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 6 it suffices to prove that

$\forall t' < T, v_f, s'.(\mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2\ \delta\gamma) \Downarrow_{t'}^{s'} v_f \implies \exists p'.s' + p' \leqslant p_l + p_m + n_2 \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$

This means given some $t' < T, v_f, s'$ s.t. $(\mathsf{release}\, x = e_1 \;\mathsf{in}\; e_2 \;\delta\gamma) \Downarrow_{t'}^{s'} v_f$ and we need to prove that

$\exists p'.s' + p' \leqslant p_l + p_m + n_2 \wedge (p', T - t', v_f) \in [\![\tau_2 \;\sigma\iota]\!]$ (F-R0)

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}.p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in [\![(\Omega_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in [\![(\Omega_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1} + p_{m1}, T, e_1 \;\delta\gamma) \in [\![[n_1]\,\tau_1 \;\sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it means we have

$\forall t_1 < T .(e_1) \;\delta\gamma \Downarrow_{t_1} v_1 \implies (p_{m1} + p_{l1}, T - t_1, v_1) \in [\![[n_1]\,\tau_1 \;\sigma\iota]\!]$

Since we know that $(\mathsf{release}\, x = e_1 \;\mathsf{in}\; e_2) \;\delta\gamma \Downarrow - \Downarrow_{t'}^{-} v_f$ therefore from E-rel we know that $\exists t_1 < t'.(e_1) \;\delta\gamma \Downarrow_{t_1} v_1$. This means we have

$(p_{m1} + p_{l1}, T - t_1, v_1) \in [\![[n_1]\,\tau_1 \;\sigma\iota]\!]$

This means from Definition 6 we have

$\exists p_1'.p_1' + n_1 \leqslant p_{l1} + p_{m1} \wedge (p_1', T - t_1, v_1) \in [\![\tau_1]\!]$ (F-R1)

<u>IH2</u>

$(p_{l2} + p_{m2} + p_1', T - t_1, e_2 \;\delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_1 + n_2)\,\tau_2 \;\sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it means we have

$\forall t_2 < T - t_1.(e_2) \;\delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \;\delta\gamma \cup \{x \mapsto v_1\} \implies (p_{m2} + p_{l2} + p_1' + n_2, T - t_1 - t_2, (e_2) \;\delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_1 + n_2)\,\tau_2 \;\sigma\iota]\!]$

Since we know that $(\mathsf{release}\, x = e_1 \;\mathsf{in}\; e_2) \;\delta\gamma \Downarrow - \Downarrow_{t'}^{-} v_f$ therefore from E-rel we know that

$\exists t_2 < t - t_1.(e_2) \;\delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \;\delta\gamma \cup \{x \mapsto v_1\}$. This means we have

$(p_{m2} + p_{l2} + p_1' + n_2, T - t_1 - t_2, (e_2) \;\delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_1 + n_2)\,\tau_2 \;\sigma\iota]\!]$

This means from Definition 6 we are given that

$\forall t_2' < T - t_1 - t_2.(e_2 \;\delta\gamma \cup \{x \mapsto v_1\}) \Downarrow_{t_2'}^{s_2} v_2 \implies \exists p_2'.s_2 + p_2' \leqslant p_{l2} + p_{m2} + p_1' + n_1 + n_2 \wedge (p_2', T - t_1 - t_2 - t_2', v_2) \in [\![\tau_2 \;\sigma\iota]\!]$

Since we know that $(\mathsf{release}\, x = e_1 \;\mathsf{in}\; e_2) \;\delta\gamma \Downarrow - \Downarrow_{t'}^{-} v_f$ therefore from E-rel we know that $\exists t_2'.(e_2) \;\delta\gamma \cup \{x \mapsto v_1\} \Downarrow^{s_2} v_2$ s.t. $t_2' = t' - t_1 - t_2 - 1$

Since $t_2' = t' - t_1 - t_2 < T - t_1 - t_2 - 1 < T - t_1 - t_2$, therefore we have

$\exists p_2'.s_2 + p_2' \leqslant p_{l2} + p_{m2} + p_1' + n_1 + n_2 \wedge (p_2', T - t_1 - t_2 - t_2', v_2) \in [\![\tau_2 \;\sigma\iota]\!]$ (F-R2)

In order to prove (F-R0) we choose p' as p_2' and it suffices to prove

(a) $s' + p'_2 \leqslant p_l + p_m + n_2$:
Since from (F-R2) we know that
$s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p'_1 + n_1 + n_2$

Since from (F-R1) we know that
$p'_1 + n_1 \leqslant p_{l1} + p_{m1}$

therefore we also have
$s_2 + p'_2 \leqslant p_{l2} + p_{m2} + p_{l1} + p_{m1} + n_2$

And finally since we know that $s' = s_2$, $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get the desired

(b) $(p'_2, T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$:
From E-rel we know that $v_f = v_2$ therefore we get the desired from (F-R2) and Lemma 69

37. T-store:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{store}\, e : \mathbb{M}\, n\, ([n]\, \tau)} \text{T-store}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(p_m, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$
To prove: $(p_l + p_m, T, \mathsf{store}\, e\ \delta\gamma) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 it suffices to prove that
$(\mathsf{store}\, e)\ \delta\gamma \Downarrow (\mathsf{store}\, e)\ \delta\gamma \implies (p_m + p_l, T, (\mathsf{store}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]$

It suffices to prove that
$(p_m + p_l, T, (\mathsf{store}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that
$\forall t' < T, v_f, n'.(\mathsf{store}\, e)\ \delta\gamma \Downarrow_{t'}^{n'} v_f \implies \exists p'.n' + p' \leqslant p_l + p_m + n \wedge (p', T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]$

This means given some $t' < T, v_f, n'$ s.t. $(\mathsf{store}\, e)\ \delta\gamma \Downarrow_{t'}^{n'} v_f$ it suffices to prove that
$\exists p'.n' + p' \leqslant p_l + p_m + n \wedge (p', T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]$

From (E-store) we know that $n' = 0$ therefore we choose p' as $p_l + p_m + n$ and it suffices to prove that
$(p_l + p_m + n, T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]_{\mathcal{E}}$

This further means that from Definition 6 we have
$\exists p''.p'' + n \leqslant p_l + p_m + n \wedge (p'', T - t', v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}\}$

We choose p'' as $p_l + p_m$ and it suffices to prove that
$(p_l + p_m, T - t', v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}\}$ (F-S0)

<u>IH</u>

$(p_l + p_m, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we have

$\forall t_1 < T\ .(e)\ \delta\gamma \Downarrow_{t_1} v_f \implies (p_m + p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we know that $(\mathsf{store}\, e)\ \delta\gamma \Downarrow - \Downarrow^0_{t'} v_f$ therefore from (E-store) we know that $\exists t_1 < t'.e\ \delta\gamma \Downarrow_{t_1} v_f$ where $t_1 + 1 = t'$

Therefore from Lemma 69 we get $(p_m + p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ and we are done

□

Lemma 13 (Γ Subtyping: domain containment). $\forall p, \gamma, \Gamma_1, \Gamma_2.$
$\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2 \implies \forall x : \tau \in \Gamma_2.\ x : \tau' \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: .}\ \text{sub-lBase}$$

   To prove: $\forall x : \tau' \in (.).x : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

   Trivial

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta; \Delta \vdash \tau' <: \tau \qquad \Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2, x : \tau}\ \text{sub-lBase}$$

   To prove: $\forall y : \tau \in \Gamma_2.y : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

   This means given some $y : \tau \in (\Gamma_2, x : \tau)$ it suffices to prove that

   $y : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

   The follwing cases arise:

   - $y = x$:
     In this case we are given that $x : \tau' \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$
     Therefore we are done
   - $y \neq x$:
     Since we are given that $\Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2$ therefore we get the desired from IH

□

Lemma 14 (Ω Subtyping: domain containment). $\forall p, \gamma, \Omega_1, \Omega_2.$
$\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2 \implies$
$\forall x :_{a<I} \tau \in \Omega_2.\ x :_{a<J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leqslant J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Omega <: .} \text{sub-mBase}$$

To prove: $\forall x :_{a<I} \tau \in (.).x :_{a<J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leqslant J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$
Trivial

2. sub-lInd:

$$\frac{\Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau \qquad \begin{array}{c} x :_{a<J} \tau' \in \Omega_1 \\ \Theta; \Delta \vdash I \leqslant J \end{array} \qquad \Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2, x :_{a<I} \tau} \text{sub-mInd}$$

To prove: $\forall y :_{a<I} \tau \in \Omega_2.y :_{a<J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leqslant J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$

This means given some $y :_{a<I} \tau \in (\Omega_2, x :_{a<I} \tau)$ it suffices to prove that
$y :_{a<J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leqslant J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$
The follwing cases arise:

- $y = x$:
  In this case we are given that
  $x :_{a<J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leqslant J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$
  Therefore we are done
- $y \neq x$:
  Since we are given that $\Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2$ therefore we get the desired from IH

□

Lemma 15 (Γ subtyping lemma)**.** $\forall p, \gamma, \Gamma_1, \Gamma_2, \sigma, \iota.$
$\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2 \implies [\![\Gamma_1 \sigma \iota]\!] \subseteq [\![\Gamma_2 \sigma \iota]\!]$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma <: .} \text{sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in [\![\Gamma_1 \sigma \iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Gamma_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$
From Definition 7 it suffices to prove that
$\exists f : \mathcal{V}ars \to \mathcal{P}ots. (\forall x \in dom(.). (f(x), T, \gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}}) \wedge (\sum_{x \in dom(.)} f(x) \leqslant p)$
We choose f as a constant function $f' - = 0$ and we get the desired

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta; \Delta \vdash \tau' <: \tau \qquad \Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2, x : \tau} \text{sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in [\![\Gamma_1 \sigma \iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![\Gamma_2, x : \tau]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Gamma_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![\Gamma_2, x : \tau]\!]_{\mathcal{E}}$

This means from Definition 7 we are given that
$\exists f : \mathcal{V}ars \to \mathcal{P}ots.$
$(\forall x \in dom(\Gamma_1). (f(x), T, \gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}})$ (L0)
$(\sum_{x \in dom(\Gamma_1)} f(x) \leqslant p)$ (L1)

Similarly from Definition 7 it suffices to prove that
$\exists f' : \mathcal{V}ars \to \mathcal{P}ots. (\forall y \in dom(\Gamma_2, x : \tau). (f'(y), T, \gamma(y)) \in [\![\Gamma(y)]\!]_{\mathcal{E}}) \wedge (\sum_{y \in dom(\Gamma_2, x:\tau)} f'(y) \leqslant p)$

We choose f' as f and it suffices to prove that

(a) $\forall y \in dom(\Gamma_2, x : \tau). (f(y), T, \gamma(y)) \in [\![\Gamma(y)]\!]_{\mathcal{E}}$:
This means given some $y \in dom(\Gamma_2, x : \tau)$ it suffices to prove that
$(f(y), T, \gamma(y)) \in [\![\tau_2]\!]_{\mathcal{E}}$ where say $\Gamma(y) = \tau_2$

From Lemma 13 we know that
$y : \tau_1 \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2$

By instantiating (L0) with the given y
$(f(y), T, \gamma(y)) \in [\![\tau_1]\!]_{\mathcal{E}}$

Finally from Lemma 18 we also get $(f(y), T, \gamma(y)) \in [\![\tau_2]\!]_{\mathcal{E}}$
And we are done

(b) $(\sum_{y \in dom(\Gamma_2, x:\tau)} f(y) \leqslant p)$:
From (L1) we know that $(\sum_{x \in dom(\Gamma_1)} f(x) \leqslant p)$ and since from Lemma 13 we know that $dom(\Gamma_2, x : \tau) \subseteq dom(\Gamma_1)$ therefore we also have
$(\sum_{y \in dom(\Gamma_2, x:\tau)} f(y) \leqslant p)$

□

Lemma 16 (Ω subtyping lemma)**.** $\forall p, \gamma, \Omega_1, \Omega_2, \sigma, \iota.$
$\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2 \implies [\![\Omega_1 \sigma \iota]\!] \subseteq [\![\Omega_2 \sigma \iota]\!]$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Omega <: .} \text{sub-mBase}$$

To prove: $\forall (p, T, \gamma) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$

From Definition 7 it suffices to prove that

$\exists f : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots. (\forall (x :_{a<I} \tau) \in .. \forall 0 \leqslant i < I. (f\ x\ i, T, \delta(x)) \in [\![\tau[i/a]]\!]_{\mathcal{E}}) \wedge (\sum_{x:_{a<I}\tau \in .} \sum_{0 \leqslant i < I} f\ x\ i) \leqslant p$

We choose f as a constant function $f' - = 0$ and we get the desired

2. sub-lInd:

$$\frac{\Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau \qquad \begin{array}{c} x :_{a<J} \tau' \in \Omega_1 \\ \Theta; \Delta \vdash I \leqslant J \end{array} \qquad \Psi; \Theta; \Delta \vdash \Omega_1 / x <: \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2, x :_{a<I} \tau} \text{ sub-mInd}$$

To prove: $\forall (p, T, \gamma) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![\Omega_2, x : \tau]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![\Omega_2, x : \tau]\!]_{\mathcal{E}}$

This means from Definition 7 we are given that

$\exists f : \mathcal{V}ars \to \mathcal{P}ots.$

$(\forall (x :_{a<I} \tau) \in \Omega_1. \forall 0 \leqslant i < I. (f\ x\ i, T, \delta(x)) \in [\![\tau[i/a]]\!]_{\mathcal{E}})$ (L0)

$(\sum_{x:_{a<I}\tau \in \Omega_1} \sum_{0 \leqslant i < I} f\ x\ i) \leqslant p$ (L1)

Similarly from Definition 7 it suffices to prove that

$\exists f' : \mathcal{V}ars \to \mathcal{I}ndices \to \mathcal{P}ots. (\forall (y :_{a<I_y} \tau_y) \in \Omega_2, x : \tau. \forall 0 \leqslant i < I_y. (f\ x\ i, T, \delta(y)) \in [\![\tau_y[i/a]]\!]_{\mathcal{E}}) \wedge (\sum_{y:_{a<I_y}\tau \in \Omega_2, x:\tau} \sum_{0 \leqslant i < I_y} f'\ y\ i) \leqslant p$

We choose f' as f and it suffices to prove that

(a) $(\forall (y :_{a<I_y} \tau_y) \in \Omega_2, x : \tau. \forall 0 \leqslant i < I_y. (f\ x\ i, T, \delta(y)) \in [\![\tau_y[i/a]]\!]_{\mathcal{E}})$:
 This means given some $(y :_{a<I} \tau_y) \in \Omega_2, x : \tau$ and some $0 \leqslant i < I_y$ it suffices to prove that
 $(f\ x\ i, T, \delta(y)) \in [\![\tau_y[i/a]]\!]_{\mathcal{E}})$

 From Lemma 13 we know that
 $y :_{a<J_y} \tau_1 \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I_y \leqslant J_y \wedge \Psi; \Theta, a; \Delta, a < I_y \vdash \tau_1 <: \tau_y$

 Instantiating (L0) with the given y and i we get
 $(f\ x\ i, T, \delta(y)) \in [\![\tau_1[i/a]]\!]_{\mathcal{E}}$

 Finally using Lemma 18 we also get
 $(f\ x\ i, T, \delta(y)) \in [\![\tau_y[i/a]]\!]_{\mathcal{E}}$

(b) $(\sum_{y:_{a<I_y}\tau_y \in \Omega_2, x:\tau} \sum_{0 \leqslant i < I_y} f'\ y\ i) \leqslant p$:
 From Lemma 14 we know that
 $\forall y :_{a<I_y} \tau_y \in (\Omega_2, x : \tau). y :_{a<J_y} \tau_1 \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I_y \leqslant J_y \wedge \Psi; \Theta, a; \Delta, a < I_y \vdash \tau_1 <: \tau_y$
 And since from (L1) we know that $(\sum_{x:_{a<I}\tau \in \Omega_1} \sum_{0 \leqslant i < I} f\ x\ i) \leqslant p$ therefore we also have
 $(\sum_{y:_{a<I_y}\tau_y \in \Omega_2, x:\tau} \sum_{0 \leqslant i < I_y} f'\ y\ i) \leqslant p$

$\square$

Lemma 17 (Value subtyping lemma)**.** $\forall \Psi, \Theta, \Delta, \tau \in Type, \tau', \sigma, \iota.$
$\Psi; \Theta; \Delta \vdash \tau <: \tau' \wedge . \models \Delta\iota \implies [\![\tau\ \sigma\iota]\!] \subseteq [\![\tau'\ \sigma\iota]\!]$

Proof. Proof by induction on the $\Psi; \Theta; \Delta \vdash \tau <: \tau'$ relation

1. sub-refl:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{ sub-refl}$$

   To prove: $\forall (p, T, v) \in [\![\tau\ \sigma\iota]\!] \implies (p, T, v) \in [\![\tau\ \sigma\iota]\!]$

   Trivial

2. sub-arrow:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1 \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau_1' \multimap \tau_2'} \text{ sub-arrow}$$

   To prove: $\forall (p, T, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!] \implies (p, T, \lambda x.e) \in [\![(\tau_1' \multimap \tau_2')\ \sigma\iota]\!]$

   This means given some $(p, T, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$ we need to prove $(p, T, \lambda x.e) \in [\![(\tau_1' \multimap \tau_2')\ \sigma\iota]\!]$

   From Definition 6 we are given that
   $\forall p', e', T'{<}T\ .(p', T', e') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SL0)

   Also from Definition 6 it suffices to prove that
   $\forall p', e', T''{<}T\ .(p', T'', e') \in [\![\tau_1'\ \sigma\iota]\!]_{\mathcal{E}} \implies (p + p', T'', e[e'/x]) \in [\![\tau_2'\ \sigma\iota]\!]_{\mathcal{E}}$

   This means given some p', e', T'' s.t $(p', T'', e') \in [\![\tau_1'\ \sigma\iota]\!]_{\mathcal{E}}$ we need to prove $(p + p', T'', e[e'/x]) \in [\![\tau_2'\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SL1)

   Since $\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1$ therefore from Lemma 18 we know that given some $(p', T'', e'') \in [\![\tau_1'\ \sigma\iota]\!]$ we also have $(p', T'', e'') \in [\![\tau_1\ \sigma\iota]\!]$

   Therefore instantiating (F-SL0) with p', e'', T'' we get
   $(p + p', T'', e[e''/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

   From Lemma 18 we get the desired

3. sub-tensor:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau_1' \otimes \tau_2'} \text{ sub-tensor}$$

   To prove: $\forall (p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1' \otimes \tau_2')\ \sigma\iota]\!]$

   This means given $(p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$

It suffices prove that
$(p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1' \otimes \tau_2')\ \sigma\iota]\!]$

This means from Definition 6 we are given that
$\exists p_1, p_2 . p_1 + p_2 \leqslant p \wedge (p_1, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p_2, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$

Also from Definition 6 it suffices to prove that
$\exists p_1', p_2' . p_1' + p_2' \leqslant p \wedge (p_1', T, v_1) \in [\![\tau_1'\ \sigma\iota]\!] \wedge (p_2', T, v_2) \in [\![\tau_2'\ \sigma\iota]\!]$

<u>IH1</u> $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_1')\ \sigma\iota]\!]$
<u>IH2</u> $[\![(\tau_2)\ \sigma\iota]\!] \subseteq [\![(\tau_2')\ \sigma\iota]\!]$

Instantiating p_1', p_2' with p_1, p_2 we get the desired from IH1 and IH2

4. sub-with:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \ \&\ \tau_2 <: \tau_1' \ \&\ \tau_2'} \text{ sub-with}$$

To prove: $\forall (p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1' \ \&\ \tau_2')\ \sigma\iota]\!]$

This means given $(p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]$
It suffices prove that
$(p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1' \ \&\ \tau_2')\ \sigma\iota]\!]$

This means from Definition 6 we are given that
$(p, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-SW0)

Also from Definition 6 it suffices to prove that
$(p, T, v_1) \in [\![\tau_1'\ \sigma\iota]\!] \wedge (p, T, v_2) \in [\![\tau_2'\ \sigma\iota]\!]$

<u>IH1</u> $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_1')\ \sigma\iota]\!]$
<u>IH2</u> $[\![(\tau_2)\ \sigma\iota]\!] \subseteq [\![(\tau_2')\ \sigma\iota]\!]$

We get the desired from (F-SW0), IH1 and IH2

5. sub-sum:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau_1' \oplus \tau_2'} \text{ sub-sum}$$

To prove: $\forall (p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle v_1, v_2 \rangle) \in [\![(\tau_1' \oplus \tau_2')\ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$
It suffices prove that
$(p, T, v) \in [\![(\tau_1' \oplus \tau_2')\ \sigma\iota]\!]$

This means from Definition 6 2 cases arise

(a) $v = \mathsf{inl}(v')$:
This means from Definition 6 we have $(p, T, v') \in \llbracket \tau_1\ \sigma\iota \rrbracket$ (F-SS0)

Also from Definition 6 it suffices to prove that
$(p, T, v') \in \llbracket \tau_1'\ \sigma\iota \rrbracket$

<u>IH</u> $\llbracket (\tau_1)\ \sigma\iota \rrbracket \subseteq \llbracket (\tau_1')\ \sigma\iota \rrbracket$

We get the desired from (F-SS0), IH

(b) $v = \mathsf{inr}(v')$:
Symmetric reasoning as in the inl case

6. sub-potential:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Psi; \Theta; \Delta \vdash n' \leqslant n}{\Psi; \Theta; \Delta \vdash [n]\,\tau <: [n']\,\tau'} \text{ sub-potential}$$

To prove: $\forall (p, T, v) \in \llbracket [n]\,\tau\ \sigma\iota \rrbracket.(p, T, v) \in \llbracket [n']\,\tau'\ \sigma\iota \rrbracket$

This means given $(p, T, v) \in \llbracket [n]\,\tau\ \sigma\iota \rrbracket$ and we need to prove
$(p, T, v) \in \llbracket [n']\,\tau'\ \sigma\iota \rrbracket$

This means from Definition 6 we are given
$\exists p'.p' + n \leqslant p \wedge (p', T, v) \in \llbracket \tau\ \sigma\iota \rrbracket$ (F-SP0)

And we need to prove
$\exists p''.p'' + n' \leqslant p \wedge (p'', T, v) \in \llbracket \tau'\ \sigma\iota \rrbracket$ (F-SP1)

In order to prove (F-SP1) we choose p'' as p'

Since from (F-SP0) we know that $p' + n \leqslant p$ and we are given that $n' \leqslant n$ therefore we also have $p' + n' \leqslant p$

<u>IH</u> $(p', T, v) \in \llbracket \tau'\ \sigma\iota \rrbracket$

$(p', T, v) \in \llbracket \tau'\ \sigma\iota \rrbracket$ we get directly from IH

7. sub-monad:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Psi; \Theta; \Delta \vdash n \leqslant n'}{\Psi; \Theta; \Delta \vdash \mathbb{M}\,n\,\tau <: \mathbb{M}\,n'\,\tau'} \text{ sub-monad}$$

To prove: $\forall (p, T, v) \in \llbracket \mathbb{M}\,n\,\tau\ \sigma\iota \rrbracket.(p, T, v) \in \llbracket \mathbb{M}\,n'\,\tau'\ \sigma\iota \rrbracket$

This means given $(p, T, v) \in \llbracket \mathbb{M}\,n\,\tau\ \sigma\iota \rrbracket$ and we need to prove
$(p, T, v) \in \llbracket \mathbb{M}\,n'\,\tau'\ \sigma\iota \rrbracket$

This means from Definition 6 we are given
$\forall t' < T, n_1, v'.v \Downarrow_{t'}^{n_1} v' \implies \exists p'.n_1 + p' \leqslant p + n \wedge (p', T - t', v') \in \llbracket \tau\ \sigma\iota \rrbracket$ (F-SM0)

Again from Definition 6 we need to prove that
$\forall t'' < T, n_2, v''.v \Downarrow_{t''}^{n_2} v'' \implies \exists p''.n_1 + p'' \leqslant p + n' \wedge (p'', T - t'', v') \in [\![\tau' \ \sigma\iota]\!]$

This means given some $t'' < T, n_2, v''$ s.t. $v \Downarrow_{t''}^{n_2} v''$ it suffices to prove that
$\exists p''.n_1 + p'' \leqslant p + n' \wedge (p'', T - t'', v') \in [\![\tau' \ \sigma\iota]\!]$ (F-SM1)

Instantiating (F-SM0) with t'', n_2, v'' Since $v \Downarrow_{t''}^{n_2} v''$ therefore from (F-SM0) we know that
$\exists p'.n_1 + p' \leqslant p + n \wedge (p', T - t'', v'') \in [\![\tau \ \sigma\iota]\!]$ (F-SM2)

<u>IH</u> $[\![\tau \ \sigma\iota]\!] \subseteq [\![\tau' \ \sigma\iota]\!]$

In order to prove (F-SM1) we choose p'' as p' and we need to prove

(a) $n_1 + p' \leqslant p + n'$:
Since we are given that $n \leqslant n'$ therefore we get the desired from (F-SM2)

(b) $(p', T - t'', v') \in [\![\tau' \ \sigma\iota]\!]$
We get this directly from IH

8. sub-subExp:

$$\frac{\Psi; \Theta, a; \Delta, a < J \vdash \tau <: \tau' \qquad \Psi; \Theta, a; \Delta \vdash J \leqslant I}{\Psi; \Theta; \Delta \vdash !_{a<I}\tau <: !_{a<J}\tau'} \text{ sub-subExp}$$

To prove: $\forall (p, T, v) \in [\![!_{a<I}\tau \ \sigma\iota]\!].(p, T, v) \in [\![!_{a<J}\tau' \ \sigma\iota]\!]$

This means given $(p, T, !v) \in [\![!_{a<I}\tau \ \sigma\iota]\!]$ and we need to prove
$(p, T, !v) \in [\![!_{a<J}\tau' \ \sigma\iota]\!]$

This means from Definition 6 we are given
$\exists p_0, \ldots, p_{I-1}.p_0 + \ldots + p_{I-1} \leqslant p \wedge \forall 0 \leqslant i < I.(p_i, T, v) \in [\![\tau[i/a]]\!]$ (F-SE0)

Again from Definition 6 we need to prove that
$\exists p'_0, \ldots, p'_{J-1}.p'_0 + \ldots + p'_{J-1} \leqslant p \wedge \forall 0 \leqslant j < J.(p_j, T, v) \in [\![\tau'[j/a]]\!]$ (F-SE1)

In order to prove (F-SE1) we choose $p'_0 \ldots p'_{J-1}$ as $p_0 \ldots p_{J-1}$ and we need to prove

(a) $p_0 + \ldots + p_{J-1} \leqslant p$:
Since we are given that $J \leqslant I$ therefore we get the desired from (F-SE0)

(b) $\forall 0 \leqslant j < J.(p_j, T, v) \in [\![\tau'[j/a] \ \sigma\iota]\!]$
We get this directly from IH and (F-SE0)

9. sub-list:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n \ \tau <: L^n \ \tau'} \text{ sub-list}$$

To prove: $\forall (p, T, v) \in [\![L^n \ \tau \ \sigma\iota]\!].(p, T, v) \in [\![L^n \ \tau' \ \sigma\iota]\!]$

This means given $(p, T, v) \in \llbracket L^n\ \tau\ \sigma\iota \rrbracket$ and we need to prove
$(p, T, v) \in \llbracket L^n\ \tau'\ \sigma\iota \rrbracket$

We induct on $(p, T, v) \in \llbracket L^n\ \tau\ \sigma\iota \rrbracket$

(a) $(p, T, nil) \in \llbracket L^0\ \tau\ \sigma\iota \rrbracket$:
We need to prove $(p, T, nil) \in \llbracket L^0\ \tau'\ \sigma\iota \rrbracket$
We get this directly from Definition 6

(b) $(p, T, v' :: l') \in \llbracket L^{m+1}\ \tau\ \sigma\iota \rrbracket$:
In this case we are given $(p, T, v' :: l') \in \llbracket L^{m+1}\ \tau\ \sigma\iota \rrbracket$
and we need to prove $(p, T, v' :: l') \in \llbracket L^{m+1}\ \tau'\ \sigma\iota \rrbracket$

This means from Definition 6 are given
$\exists p_1, p_2.p_1 + p_2 \leqslant p \wedge (p_1, T, v') \in \llbracket \tau\ \sigma\iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau\ \sigma\iota \rrbracket$ (Sub-List0)

Similarly from Definition 6 we need to prove that
$\exists p_1', p_2'.p_1' + p_2' \leqslant p \wedge (p_1', T, v') \in \llbracket \tau'\ \sigma\iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau'\ \sigma\iota \rrbracket$

We choose p_1' as p_1 and p_2' as p_2 and we get the desired from (Sub-List0) IH of outer induction and IH of innner induction

10. sub-exist:

$$\frac{\Psi; \Theta, s; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s.\tau <: \exists s.\tau'} \text{ sub-exist}$$

To prove: $\forall (p, T, v) \in \llbracket \exists s.\tau\ \sigma\iota \rrbracket.(p, T, v) \in \llbracket \exists s.\tau'\ \sigma\iota \rrbracket$

This means given some $(p, T, v) \in \llbracket \exists s.\tau\ \sigma\iota \rrbracket$ we need to prove
$(p, T, v) \in \llbracket \exists s.\tau'\ \sigma\iota \rrbracket$

From Definition 6 we are given that
$\exists s'.(p, T, v) \in \llbracket \tau\sigma\iota[s'/s] \rrbracket$ (F-exist0)

<u>IH</u>: $\llbracket (\tau)\ \sigma\iota \cup \{s \mapsto s'\} \rrbracket \subseteq \llbracket (\tau')\ \sigma\iota \cup \{s \mapsto s'\} \rrbracket$

Also from Definition 6 it suffices to prove that
$\exists s''.(p, T, v) \in \llbracket \tau'\sigma\iota[s''/s] \rrbracket$

We choose s'' as s' and we get the desired from IH

11. sub-typePoly:

$$\frac{\Psi, \alpha; \Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall\alpha.\tau_1 <: \forall\alpha.\tau_2} \text{ sub-typePoly}$$

To prove: $\forall (p, T, \Lambda\alpha.e) \in \llbracket (\forall\alpha.\tau_1)\ \sigma\iota \rrbracket.(p, T, \Lambda\alpha.e) \in \llbracket (\forall\alpha.\tau_2)\ \sigma\iota \rrbracket$

This means given some $(p, T, \Lambda\alpha.e) \in \llbracket (\forall\alpha.\tau_1)\ \sigma\iota \rrbracket$ we need to prove

$(p, T, \Lambda\alpha.e) \in [\![(\forall\alpha.\tau_2)\ \sigma\iota]\!]$

From Definition 6 we are given that
$\forall\tau', T'<T\ .(p, T', e) \in [\![\tau_1[\tau'/\alpha]]\!]_{\mathcal{E}}$ (F-STF0)

Also from Definition 6 it suffices to prove that
$\forall\tau'', T''<T\ .(p, T''\ e) \in [\![\tau_2[\tau''/\alpha]]\!]_{\mathcal{E}}$

This means given some $\tau'', T''<T$ and we need to prove
$(p, T'', e[\tau''/\alpha]) \in [\![\tau_2[\tau''/\alpha]]\!]_{\mathcal{E}}$ (F-STF1)

<u>IH</u>: $[\![(\tau_1)\ \sigma \cup \{\alpha \mapsto \tau''\}\iota]\!] \subseteq [\![(\tau_2)\ \sigma \cup \{\alpha \mapsto \tau''\}\iota]\!]$

Instantiating (F-STF0) with τ'', T'' we get
$(p, T'', e) \in [\![\tau_1[\tau''/\alpha]]\!]_{\mathcal{E}}$

and finally from IH we get the desired

12. sub-indexPoly:

$$\frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i.\tau_1 <: \forall i.\tau_2}\ \text{sub-indexPoly}$$

To prove: $\forall(p, T, \Lambda i.e) \in [\![(\forall i.\tau_1)\ \sigma\iota]\!].(p, T, \Lambda i.e) \in [\![(\forall i.\tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \Lambda i.e) \in [\![(\forall i.\tau_1)\ \sigma\iota]\!]$ we need to prove
$(p, T, \Lambda i.e) \in [\![(\forall i.\tau_2)\ \sigma\iota]\!]$

From Definition 6 we are given that
$\forall I, T'<T\ .(p, T', e) \in [\![\tau_1[I/i]]\!]_{\mathcal{E}}$ (F-SIF0)

Also from Definition 6 it suffices to prove that
$\forall I', T''<T\ .(p, T'', e) \in [\![\tau_2[I'/i]]\!]_{\mathcal{E}}$

This means given some $I', T''<T$ and we need to prove
$(p, T'', e) \in [\![\tau_2[I'/i]]\!]_{\mathcal{E}}$ (F-SIF1)

<u>IH</u>: $[\![(\tau_1)\ \sigma\iota \cup \{i \mapsto I'\}]\!] \subseteq [\![(\tau_2)\ \sigma\iota \cup \{i \mapsto I'\}]\!]$

Instantiating (F-SIF0) with I', T'' we get
$(p, T'', e) \in [\![\tau_1[I'/i]]\!]_{\mathcal{E}}$

and finally from IH we get the desired

13. sub-constraint:

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_2 \qquad \Theta;\Delta \models c_2 \implies c_1}{\Psi;\Theta;\Delta \vdash c_1 \Rightarrow \tau_1 <: c_2 \Rightarrow \tau_2} \text{sub-constraint}$$

To prove: $\forall (p, T, \Lambda.e) \in [\![(c_1 \Rightarrow \tau_1)\ \sigma\iota]\!].(p, T, \Lambda.e) \in [\![(c_2 \Rightarrow \tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \Lambda.e) \in [\![(c_1 \Rightarrow \tau_1)\ \sigma\iota]\!]$ we need to prove $(p, T, \Lambda.e) \in [\![(c_2 \Rightarrow \tau_2)\ \sigma\iota]\!]$

From Definition 6 we are given that
$\forall\ T'{<}T\ .\models c_1\iota \implies (p, T', e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SC0)

Also from Definition 6 it suffices to prove that
$\forall\ T''{<}T\ .\models c_2\iota \implies (p, T'', e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$

This means given some $T''{<}T$ s.t. $\models c_2\iota$ and we need to prove
$(p, T'', e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$ (F-SC1)

Since we are given that $\Theta;\Delta \models c_2 \implies c_1$ therefore we know that $.\models c_1\iota$

Hence from (F-SC0) we have
$(p, T'', e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SC2)

<u>IH</u>: $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_2)\ \sigma\iota]\!]$

Therefore we ge the desired from IH and (F-SC2)

14. sub-CAnd:

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_2 \qquad \Theta;\Delta \models c_1 \implies c_2}{\Psi;\Theta;\Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{sub-CAnd}$$

To prove: $\forall (p, T, v) \in [\![(c_1 \& \tau_1)\ \sigma\iota]\!].(p, T, v) \in [\![(c_2 \& \tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, v) \in [\![(c_1 \& \tau_1)\ \sigma\iota]\!]$ we need to prove $(p, T, v) \in [\![(c_2 \& \tau_2)\ \sigma\iota]\!]$

From Definition 6 we are given that
$.\models c_1\iota \wedge (p, T, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SCA0)

Also from Definition 6 it suffices to prove that
$.\models c_2\iota \wedge (p, T, e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $\Theta;\Delta \models c_2 \implies c_1$ and $.\models c_1\iota$ therefore we also know that $.\models c_2\iota$

Also from (F-SCA0) we have $(p, T, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SCA1)

<u>IH</u>: $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_2)\ \sigma\iota]\!]$

Therefore we ge the desired from IH and (F-SCA1)

15. sub-potArrow:

$$\frac{\Psi; \Theta; \Delta \vdash k'}{\Psi; \Theta; \Delta \vdash [k](\tau_1 \multimap \tau_2) <: ([k']\, \tau_1 \multimap [k' + k]\, \tau_2)} \text{ sub-potArrow}$$

To prove: $\forall (p, T, \lambda x.e) \in [\![([k](\tau_1 \multimap \tau_2))\ \sigma\iota]\!].(p, T, \lambda x.e) \in [\![([k']\, \tau_1 \multimap [k' + k]\, \tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \lambda x.e) \in [\![([k](\tau_1 \multimap \tau_2))\ \sigma\iota]\!]$ we need to prove $(p, T, \lambda x.e) \in [\![(([k']\, \tau_1 \multimap [k' + k]\, \tau_2))\ \sigma\iota]\!]$

From Definition 6 we are given that
$\exists p'.p' + k \leqslant p \wedge (p', T, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]\}$ (F-SPA0)

Again from Definition 6 we know that
$\forall p''', e', T' {<} T\ .(p''', T', e') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p' + p''', T', e[e'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SPA1)

Also from Definition 6 it suffices to prove that
$\forall p'', e'', T'' {<} T\ .(p'', T'', e'') \in [\![[k']\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p + p'', T'', e[e''/x]) \in [\![[k + k']\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $p'', e'', T'' {<} T$ s.t $(p'', T'', e'') \in [\![[k']\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}}$ we need to prove $(p + p'', T'', e[e''/x]) \in [\![[k + k']\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SSP2)

Applying Definition 6 on (F-SPA2) we get
$\forall v_f, t' {<} T''\ .e[e''/x] \Downarrow_{t'} v_f \implies (p + p'', T'' - t', v_f) \in [\![[k + k']\, \tau_2\ \sigma\iota]\!]$

This means that given some $v_f, t' {<} T''$ s.t. $e[e''/x] \Downarrow_{t'} v_f$ and we need to prove that $(p + p'', T'' - t', v_f) \in [\![[k + k']\, \tau_2\ \sigma\iota]\!]$

This means From Definition 6 it suffices to prove that
$\exists p''_2.p''_2 + (k + k') \leqslant (p + p'') \wedge (p''_2, T'' - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]\}$ (F-SPA4)

Also since we are given that $(p'', T'', e'') \in [\![[k']\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}}$ we apply Definition 6 on it to obtain
$\forall t {<} T'', v'.e'' \Downarrow_t v' \implies (p'', T'' - t, v') \in [\![[k']\, \tau_1\ \sigma\iota]\!]$
Also since we are given that $e[e''/x] \Downarrow_{t'} v_f$ therefore we also know that
$\exists t'' < t' {<} T''\ .e'' \Downarrow_{t''} v''$
Instantiating with t'', v'' we get $(p'', T'' - t'', v'') \in [\![[k']\, \tau_1\ \sigma\iota]\!]$

Again using Definition 6 we know that we are given
$\exists p''_1.p''_1 + k' \leqslant p'' \wedge (p''_1, T'' - t'', v'') \in [\![\tau_1\ \sigma\iota]\!]$ (F-SPA3)

Since $(p''_1, T'' - t'', v'') \in [\![\tau_1\ \sigma\iota]\!]$ therefore from Definition 6 we also have $(p''_1, T'' - t'', v'') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

Instantiating (F-SPA1) with $p''_1, v'', T'' - t''$ we get

$(p' + p_1'', T'' - t'', e[v''/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 6 this means that

$\forall t''' < T'' - t'', v_f . e[v''/x] \Downarrow v_f \implies (p' + p_1'', T'' - t'' - t''', v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-SPA4.1)

Since we know that $e[e''/x] \Downarrow_{t'} v_f$ therefore we also know that $\exists t''' . e[v''/x] \Downarrow_{t'''} v_f$ s.t. $t''' + t'' \leqslant t'$

Since we already know that $\exists t'' < t' < T'' . e'' \Downarrow_{t''} v''$ therefore we have $t'' + t''' \leqslant t' < T''$.

Instantiating (F-SPA4.1) with t''' we get

$(p' + p_1'', T'' - t'' - t''', v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-SPA5)

Since from (F-SPA0) we know that

$p' + k \leqslant p$

And from (F-SPA3) we know that

$p_1'' + k' \leqslant p''$

We add the two to get

$p' + p_1'' + k + k' \leqslant p + p''$ (F-SPA6)

In order to prove (F-SPA4) we choose p_2'' as $p' + p_1''$

and we get the desired from (F-SPA6) and (F-SPA5) and Lemma 69

16. sub-potZero:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: [0]\,\tau} \text{ sub-potZero}$$

To prove: $\forall (p, T, v) \in [\![\tau\ \sigma\iota]\!] . (p, T, v) \in [\![[0]\,\tau\ \sigma\iota]\!]$

This means that given $(p, T, v) \in [\![\tau\ \sigma\iota]\!]$

And we need to prove $(p, T, v) \in [\![[0]\,\tau\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that

$\exists p' . p' + 0 \leqslant p \wedge (p', T, v) \in [\![\tau\ \sigma\iota]\!]$

We choose p' as p and we get the desired

17. sub-familyAbs:

$$\frac{\Psi; \Theta, i : S \vdash \tau <: \tau'}{\Psi; \Theta \vdash \lambda_t i : S . \tau <: \lambda_t i : S . \tau'} \text{ sub-familyAbs}$$

To prove:

$\forall f \in [\![\lambda_t i : S . \tau\ \sigma\iota]\!] . f \in [\![\lambda_t i : S . \tau'\ \sigma\iota]\!]$

This means given $f \in [\![\lambda_t i : S . \tau\ \sigma\iota]\!]$ and we need to prove

$f \in [\![\lambda_t i :S .\tau' \ \sigma\iota]\!]$

This means from Definition 6 we are given

$\forall I.f \ I \in [\![\tau[I/i] \ \sigma\iota]\!]$ (F-SFAbs0)

This means from Definition 6 we need to prove

$\forall I'.f \ I' \in [\![\tau'[I'/i] \ \sigma\iota]\!]$

This further means that given some I' we need to prove

$f \ I' \in [\![\tau'[I'/i] \ \sigma\iota]\!]$ (F-SFAbs1)

Instantiating (F-SFAbs0) with I' we get

$f \ I' \in [\![\tau[I'/i] \ \sigma\iota]\!]$

From IH we know that $[\![\tau \ \sigma\iota \cup \{i \mapsto I' \ \iota\}]\!] \subseteq [\![\tau' \ \sigma\iota \cup \{i \mapsto I' \ \iota\}]\!]$

And this completes the proof.

18. Sub-tfamilyApp1:

$$\frac{}{\Psi; \Theta; \Delta \vdash \lambda_t i :S .\tau \ I <: \tau[I/i]} \text{ sub-familyApp1}$$

To prove: $\forall (p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!].(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$ and we need to prove $(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$

This means from Definition 6 we are given

$(p, T, v) \in [\![\lambda_t i :S .\tau]\!] \ I \ \sigma\iota$

This further means that we have

$(p, T, v) \in f \ I\iota$ where $f \ I = [\![\tau\sigma[I\iota/i]]\!]$

This means we have $(p, T, v) \in [\![\tau\sigma[I\iota/i]]\!]$

And this completes the proof.

19. Sub-tfamilyApp2:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: \lambda_t i :S .\tau \ I} \text{ sub-familyApp2}$$

To prove: $\forall (p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!].(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$ (Sub-tF0)

And we need to prove

$(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$

This means from Definition 6 it suffices to prove that

$(p, T, v) \in [\![\lambda_t i :S .\tau]\!] \ I \ \sigma\iota$

It further suffices to prove that

$(p, T, v) \in f\ I\iota$ where $f\ I\iota = [\![\tau\sigma[I\iota/i]]\!]$

which means we need to show that
$(p, T, v) \in [\![\tau\sigma[I\iota/i]]\!]$
We get this directly from (Sub-tF0)

20. sub-bSum:

$$\frac{}{\Psi; \Theta; \Delta \vdash [\sum_{a<I} K]\ !_{a<I}\tau <:!_{a<I}\ [K]\ \tau}\ \text{sub-bSum}$$

To prove: $\forall (p, T, v) \in [\![[\sum_{a<I} K]\ !_{a<I}\tau\ \sigma\iota]\!] \implies (p, T, v) \in [\![!_{a<I}\ [K]\ \tau\ \sigma\iota]\!]$

This means given some (p, T, v) s.t $(p, T, v) \in [\![[\sum_{a<I} K]\ !_{a<I}\tau\ \sigma\iota]\!]$ it suffices to prove that $(p, T, v) \in [\![!_{a<I}\ [K]\ \tau\ \sigma\iota]\!]$

This means from Definition 6 we are given that
$\exists p'.p' + \sum_{a<I} K \leqslant p \wedge (p', T, v) \in [\![!_{a<I}\tau\ \sigma\iota]\!]\}$ (Sub-BS0)

Since $(p', T, v) \in [\![!_{a<I}\tau\ \sigma\iota]\!]\}$ therefore again from Definition 6 it means that $\exists e'.\ v = !e'$ and
$\exists p_0, \ldots, p_{I-1}.p_0 + \ldots + p_{I-1} \leqslant p' \wedge \forall 0 \leqslant i < I.(p_i, T, e') \in [\![\tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$ (Sub-BS1)

Since $\forall 0 \leqslant i < I.(p_i, T, e') \in [\![\tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Definition 6 we have
$\forall 0 \leqslant i < I.\forall t < T, v''.e' \Downarrow_t v'' \implies (p_i, T - t, v') \in [\![\tau[i/a]\ \sigma\iota]\!]$ (Sub-BS1.1)

Since we know that $v = !e'$ therefore it suffices to prove that $(p, T, !e') \in [\![!_{a<I}\ [K]\ \tau\ \sigma\iota]\!]$

From Definition 6 it further suffices to prove that
$\exists p'_0, \ldots, p'_{I-1}.p'_0 + \ldots + p'_{I-1} \leqslant p \wedge \forall 0 \leqslant i < I.(p'_i, T, e') \in [\![[K]\ \tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$

We choose p'_0 as $p_0 + K[0/a] \ldots p'_{I-1}$ as $p_{I-1} + K[(I-1)/a]$ and it suffices to prove that

- $p'_0 + \ldots + p'_{I-1} \leqslant p$:
  We need to prove that
  $(p_0 + K[0/a]) + \ldots + (p_{I-1} + K[(I-1)/a]) \leqslant p$
  We get this from (Sub-BS0) and (Sub-BS1)
- $\forall 0 \leqslant i < I.(p'_i, T, e') \in [\![[K]\ \tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$:
  Given some $0 \leqslant i < I$ it suffices to prove that
  $(p'_i, T, e') \in [\![[K]\ \tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$

  Since p'_i is $p_i + K[i/a]$ therefore it suffices to prove that
  $(p_i + K[i/a], T, e') \in [\![[K[i/a]]\ \tau[i/a]\ \sigma\iota]\!]_{\mathcal{E}}$

  From Definition 6 we need to prove that
  $\forall v', t'' < T .e' \Downarrow_{t''} v' \implies (p_i + K[i/a], T - t'', v') \in [\![[K[i/a]]\ \tau[i/a]\ \sigma\iota]\!]$

  This means given some v' s.t $e' \Downarrow_{t''} v'$ we need to prove that

$(p_i + K[i/a], T - t'', v') \in [\![[K[i/a]]\, \tau[i/a]\ \sigma\iota]\!]$

From Definition 6 it suffices to prove that
$\exists p''. p'' + K[i/a] \leqslant p_i + K[i/a] \wedge (p'', T - t'', v') \in [\![\tau[i/a]\ \sigma\iota]\!] \}$

We choose p'' as p_i and we need to prove
$(p_i, T - t'', v') \in [\![\tau[i/a]\ \sigma\iota]\!]$
Instantiating (Sub-BS1.1) with the given i and v', t'' we get the desired

□

Lemma 18 (Expression subtyping lemma)**.** $\forall \Psi, \Theta, \Delta, \tau \in Type, \tau'.$
$\Psi; \Theta; \Delta \vdash \tau <: \tau' \implies [\![\tau\ \sigma\iota]\!]_{\mathcal{E}} \subseteq [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

Proof. To prove: $\forall (p, T, e) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}} \implies (p, T, e) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
This means given some $(p, T, e) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that
$(p, T, e) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 6 we are given
$\forall v, t < T . e \Downarrow_t v \implies (p, T - t, v) \in [\![\tau\ \sigma\iota]\!]$ (S-E0)

Similarly from Definition 6 it suffices to prove that
$\forall v', t' < T . e \Downarrow_{t'} v' \implies (p, T - t', v') \in [\![\tau'\ \sigma\iota]\!]$

This means given some $v', t' < T$ s.t $e \Downarrow_{t'} v'$ it suffices to prove that
$(p, T - t', v') \in [\![\tau'\ \sigma\iota]\!]$

Instantiating (S-E0) with v', t' we get $(p, T - t', v') \in [\![\tau\ \sigma\iota]\!]$

And finally from Lemma 17 we get the desired.

□

Theorem 19 (Soundness)**.** $\forall e, n, n', \tau \in Type, t.$
$\vdash e : \mathbb{M}\, n\, \tau \wedge e \Downarrow_t^{n'} v \implies n' \leqslant n$

Proof. From Theorem 12 we know that $(0, t+1, e) \in [\![\mathbb{M}\, n\, \tau]\!]_{\mathcal{E}}$

From Definition 6 this means we have
$\forall t' < t + 1 . e \Downarrow_{t'} v' \implies (0, t + 1 - t'v') \in [\![\mathbb{M}\, n\, \tau]\!]$

From the evaluation relation we know that $e \Downarrow_0 e$ therefore we have
$(0, t+1, e) \in [\![\mathbb{M}\, n\, \tau]\!]$

Again from Definition 6 it means we have
$\forall t'' < t + 1 . e \Downarrow_{t'}^{n'} v \implies \exists p'. n' + p' \leqslant 0 + n \wedge (p', t + 1 - t'', v) \in [\![\tau]\!]$

Since we are given that $e \Downarrow_t^{n'} v$ therefore we have
$\exists p'. n' + p' \leqslant n \wedge (p', 1, v) \in [\![\tau]\!]$

Since $p' \geqslant 0$ therefore we get $n' \leqslant n$

□

Theorem 20 (Soundness)**.** $\forall e, n, n', \tau \in Type.$
$\vdash e : [n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau \wedge e\ () \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v \implies n' \leqslant n$

Proof. From Theorem 12 we know that $(0, t_1 + t_2 + 2, e) \in \llbracket [n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau \rrbracket_{\mathcal{E}}$

Therefore from Definition 6 we know that
$\forall t' < t_1 + t_2 + 2, v.e \Downarrow_{t'} v \implies (0, t_1 + t_2 + 2 - t', v) \in \llbracket [n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau \rrbracket$ (S0)

Since we know that $e\ () \Downarrow_{t_1} -$ therefore from E-app we know that $\exists e'.e \Downarrow_{t_1} \lambda x.e'$

Instantiating (S0) with $t_1, \lambda x.e'$ we get $(0, t_2 + 2, \lambda x.e') \in \llbracket [n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau \rrbracket$

This means from Definition 6 we have
$\forall p', e', t'' < t_2 + 2.(p', t'', e'') \in \llbracket [n]\, \mathbf{1} \rrbracket_{\mathcal{E}} \implies (0 + p', t'', e'[e''/x]) \in \llbracket \mathbb{M}\, 0\, \tau \rrbracket_{\mathcal{E}}$ (S1)

Claim: $\forall t.(I, t, ()) \in \llbracket [I]\, \mathbf{1} \rrbracket_{\mathcal{E}}$
Proof:
From Definition 6 it suffices to prove that
$() \Downarrow_0 v \implies (I, t, v) \in \llbracket [I]\, \mathbf{1} \rrbracket$

Since we know that $v = ()$ therefore it suffices to prove that
$(I, t, v) \in \llbracket [I]\, \mathbf{1} \rrbracket$

From Definition 6 it suffices to prove that
$\exists p'.p' + I \leqslant I \wedge (p', t, v) \in \llbracket \mathbf{1} \rrbracket \}$

We choose p' as 0 and we get the desired

Instantiating (S1) with $n, (), t_2 + 1$ we get $(n, t_2 + 1, e'[()/x]) \in \llbracket \mathbb{M}\, 0\, \tau \rrbracket_{\mathcal{E}}$

This means again from Definition 6 we have
$\forall t' < t_2 + 1.e'[()/x] \Downarrow_{t'} v' \implies (n, t_2 + 1 - t', v') \in \llbracket \mathbb{M}\, 0\, \tau \rrbracket$

From E-val we know that $v' = e'[()/x]$ and $t' = 0$ therefore we have
$(n, t_2 + 1, e'[()/x]) \in \llbracket \mathbb{M}\, 0\, \tau \rrbracket$

Again from Definition 6 we have
$\forall t' < t_2 + 1.e'[()/x] \Downarrow_{t'}^{n'} v'' \implies \exists p'.n' + p' \leqslant n + 0 \wedge (p', t_2 + 1 - t', v'') \in \llbracket \tau \rrbracket$

Since we are given that $e \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v$ therefore we get
$\exists p'.n' + p' \leqslant n \wedge (p', 1, v'') \in \llbracket \tau \rrbracket$

Since $p' \geqslant 0$ therefore we have $n' \leqslant n$

□

1.5 Embedding dlPCF

Type translation

$$\begin{array}{lcl} (\!|b|\!) & = & b \\ (\!|[a < I]\tau_1 \multimap \tau_2|\!) & = & (!_{a<I}\, \mathbb{M}\, 0 (\!|\tau_1|\!)) \multimap [I]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!) \end{array}$$

Judgment translation

$$\boxed{\Theta; \Delta; \Gamma \vdash_K e_d : \tau \quad \rightsquigarrow \quad .; \Theta; \Delta; (\!|\Gamma|\!); . \vdash e_a : [K + count(\Gamma)]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau|\!)}$$

where

$$\begin{array}{lcl} count(.) & = & 0 \\ count(\Gamma, x : [a < I]\tau) & = & count(\Gamma) + I \end{array}$$

Definition 21 (Context translation).

$$\begin{array}{rcl} (\!|.|\!) & = & . \\ (\!|\Gamma, x : [a < I]\tau|\!) & = & (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\,0\,(\!|\tau|\!) \end{array}$$

Expression translation

$$\frac{\begin{array}{c}\Theta; \Delta \models J \geqslant 0 \\ \Theta; \Delta \models I \geqslant 1 \qquad \Theta; \Delta \vdash \sigma[0/a] <: \tau \qquad \Theta; \Delta \models [a < I]\sigma \Downarrow \qquad \Theta; \Delta \models \Gamma \Downarrow\end{array}}{\Theta; \Delta; \Gamma, x : [a < I]\sigma \vdash_J x : \tau \rightsquigarrow \lambda p.\mathsf{release} - = p \;\mathsf{in}\; \mathsf{bind} - = \uparrow^1 \mathsf{in}\; x} \text{ var}$$

$$\frac{\Theta; \Delta; \Gamma, x : [a < I]\tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\begin{array}{c}\Theta; \Delta; \Gamma \vdash_J \lambda x.e : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow \\ \lambda p_1.\,\mathsf{ret}\;\lambda y.\lambda p_2.\,\mathsf{let}\,!\,x = y \;\mathsf{in}\; \mathsf{release} - = p_1 \;\mathsf{in}\; \mathsf{release} - = p_2 \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\; a\end{array}} \text{ lam}$$

$$\frac{\begin{array}{c}\Theta; \Delta; \Gamma \vdash_J e_1 : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow e_{t1} \\ \Theta, a; \Delta, a < I; \Delta \vdash_K e_2 : \tau_1 \rightsquigarrow e_{t2} \qquad \Gamma' \sqsupseteq \Gamma \oplus \sum_{a<I} \Delta \qquad H \geqslant J + I + \sum_{a<I} K\end{array}}{\Theta; \Delta; \Gamma' \vdash_H e_1\; e_2 : \tau_2 \rightsquigarrow \lambda p.E_0} \text{ app}$$

$E_0 = \mathsf{release} - = p \;\mathsf{in}\; E_1$
$E_1 = \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; E_2$
$E_2 = \mathsf{bind}\, b = e_{t1}\; a \;\mathsf{in}\; E_3$
$E_3 = \mathsf{bind}\, c = \mathsf{store!}() \;\mathsf{in}\; E_4$
$E_4 = \mathsf{bind}\, d = \mathsf{store}() \;\mathsf{in}\; E_5$
$E_5 = b\; (coerce\; !e_{t2}\; c)\; d$

$$\frac{\begin{array}{c}\Theta, b; \Delta, b < L; \Gamma, x : [a < I]\sigma \vdash_K e : \tau \rightsquigarrow e_t \\ \tau[0/a] <: \mu \qquad \Theta, a, b; \Delta, a < I, b < L; \Gamma \vdash \tau[(b + 1 + \bigtriangleup_b^{b+1,a} I)/b] <: \sigma \\ \Gamma' \sqsubseteq \sum_{b<L} \Gamma \qquad L, M \geqslant \bigtriangleup_b^{0,1} I \qquad N \geqslant M - 1 + \sum_{b<L} K\end{array}}{\Theta; \Delta; \Gamma' \vdash_N \mathsf{fix} x.e : \mu \rightsquigarrow E_0} \text{ T-fix}$$

$E_0 = \mathsf{fix} Y.E_1$
$E_1 = \lambda p.E_2$
$E_2 = \mathsf{release} - = p \;\mathsf{in}\; E_3$
$E_3 = \mathsf{bind}\; A = \mathsf{store}() \;\mathsf{in}\; E_4$
$E_4 = \mathsf{let}\,!\,x = (E_{4.1}\; E_{4.2}) \;\mathsf{in}\; E_5$
$E_{4.1} = coerce1\; !Y$
$E_{4.2} = (\lambda u.!())\; A$
$E_5 = \mathsf{bind}\; C = \mathsf{store}() \;\mathsf{in}\; E_6$
$E_6 = e_t\; C$

1.5.1 Type preservation

Theorem 22 (Type preservation: dlPCF to λ-Amor). If $\Theta; \Delta; \Gamma \vdash_I e : \tau$ in dlPCF then there exists e' such that $\Theta; \Delta; \Gamma \vdash_I e : \tau \rightsquigarrow e'$ such that there is a derivation of $.; \Theta; \Delta; (\!|\Gamma|\!); . \vdash e' : [I + count(\Gamma)]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau|\!)$ in λ-Amor .

Proof. Proof by induction on the $\Theta; \Delta; \Gamma \vdash_I e : \tau$

- var:

$$\frac{\Theta; \Delta \models I \geqslant 1 \qquad \Theta; \Delta \vdash \sigma[0/a] <: \tau \qquad \Theta; \Delta \models [a < I]\sigma \Downarrow \qquad \Theta; \Delta \models \Gamma \Downarrow \qquad \Theta; \Delta \models J \geqslant 0}{\Theta; \Delta; \Gamma, x : [a < I]\sigma \vdash_J x : \tau \rightsquigarrow \lambda p.\mathsf{release} - = p \mathsf{~in~bind} - = \uparrow^1 \mathsf{in}~x} \text{ var}$$

D2:

$$\frac{\Theta; \Delta \vdash \sigma[0/a] <: \tau}{\Theta; \Delta \vdash (\!|\sigma[0/a]|\!) <: (\!|\tau|\!)} \text{ Lemma 27}$$

D1:

$$\frac{\dfrac{}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), \vdash x : \mathbb{M}\, 0\, (\!|\sigma|\!)[0/a]} \text{ T-var2}}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), \vdash x : \mathbb{M}\, 0\, (\!|\sigma[0/a]|\!)} \text{ Lemma 28}$$

D0:

$$\frac{\dfrac{\dfrac{}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), \vdash \uparrow^1 : \mathbb{M}\, 1\, \mathbf{1}}}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), \vdash \uparrow^1 : \mathbb{M}(I + J + count(\Gamma))\, \mathbf{1}} \qquad D1}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), \vdash \mathsf{bind} - = \uparrow^1 \mathsf{in}~x : \mathbb{M}(I + J + count(\Gamma))\, (\!|\sigma[0/a]|\!)} \text{ bind}$$

Main derivation:

$$\frac{\dfrac{\dfrac{}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!), p : ([I + J + count(\Gamma)]\, \mathbf{1}) \vdash \qquad p : ([I + J + count(\Gamma)]\, \mathbf{1})} \quad D0}{\begin{array}{c}\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!); p : ([I + J + count(\Gamma)]\, \mathbf{1}) \vdash \\ \mathsf{release} - = p \mathsf{~in~bind} - = \uparrow^1 \mathsf{in}~x : \mathbb{M}\, 0\, (\!|\tau|\!)\end{array}} \text{ T-release}}{\begin{array}{c}\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\sigma|\!); \cdot \vdash \\ \lambda p.\mathsf{release} - = p \mathsf{~in~bind} - = \uparrow^1 \mathsf{in}~x : (([I + J + count(\Gamma)]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\!|\tau|\!))\end{array}} \text{ T-lam}$$

- lam:

$$\frac{\Theta; \Delta; \Gamma, x : [a < I]\tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\begin{array}{c}\Theta; \Delta; \Gamma \vdash_J \lambda x.e : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow \\ \lambda p_1.\, \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\,!\, x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind}\, a = \mathsf{store}() \mathsf{~in~} e_t\ a\end{array}}$$

$E_0 = \lambda p_1.\, \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\,!\, x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind}\, a = \mathsf{store}() \mathsf{~in~} e_t\ a$

$E_1 = \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\,!\, x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind}\, a = \mathsf{store}() \mathsf{~in~} e_t\ a$

$E_2 = \lambda y.\lambda p_2.\, \mathsf{let}\,!\, x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind}\, a = \mathsf{store}() \mathsf{~in~} e_t\ a$

$E_3 = \lambda p_2. \, \mathsf{let}\, !\, x = y \;\mathsf{in}\; \mathsf{release} - = p_1 \;\mathsf{in}\; \mathsf{release} - = p_2 \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\ a$

$E_4 = \mathsf{let}\, !\, x = y \;\mathsf{in}\; \mathsf{release} - = p_1 \;\mathsf{in}\; \mathsf{release} - = p_2 \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\ a$

$E_{4.1} = \mathsf{release} - = p_1 \;\mathsf{in}\; \mathsf{release} - = p_2 \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\ a$

$E_{4.2} = \mathsf{release} - = p_2 \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\ a$

$E_{4.3} = \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; e_t\ a$

$T_0 = [J + count(\Gamma)]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|([a < I]\tau_1) \multimap \tau_2|\!)$

$T_{0.1} = [J + count(\Gamma)]\, \mathbf{1} \multimap \mathbb{M}\, 0((!_{a<I}\, \mathbb{M}\, 0\, (\!|\tau_1|\!)) \multimap [I]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!))$

$T_{0.2} = [J + count(\Gamma)]\, \mathbf{1}$

$T_1 = \mathbb{M}\, 0((!_{a<I}\, \mathbb{M}\, 0\, (\!|\tau_1|\!)) \multimap [I]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!))$

$T_2 = ((!_{a<I}\, \mathbb{M}\, 0\, (\!|\tau_1|\!)) \multimap [I]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!))$

$T_{2.1} = !_{a<I}\, \mathbb{M}\, 0\, (\!|\tau_1|\!)$

$T_3 = [I]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!)$

$T_{3.1} = [I]\, \mathbf{1}$

$T_4 = \mathbb{M}\, 0\, (\!|\tau_2|\!)$

$T_{4.1} = \mathbb{M}(J + I + count(\Gamma))\, \mathbf{1}$

$T_{4.2} = \mathbb{M}(J + I + count(\Gamma))\, (\!|\tau_2|\!)$

$T_{4.3} = \mathbb{M}(J + count(\Gamma))\, (\!|\tau_2|\!)$

$T_5 = [(J + I + count(\Gamma))]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\!|\tau_2|\!)$

D6:

$$\frac{}{\cdot; \Theta; \Delta; \cdot; a : [J + I + count(\Gamma)]\, \mathbf{1} \vdash a : [J + I + count(\Gamma)]\, \mathbf{1}}\ \text{var}$$

D5:

$$\frac{}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau_1|\!); \cdot \vdash e_t : T_5}\ \text{IH}$$

D4:

$$\frac{D5 \qquad D6}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau_1|\!); a : [J + I + count(\Gamma)]\, \mathbf{1} \vdash e_t\ a : T_4}\ \text{app}$$

D3:

$$\frac{\dfrac{}{\cdot; \Theta; \Delta; \cdot; \cdot \vdash \mathsf{store}() : T_{4.1}}\ \text{store} \qquad D4}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau_1|\!); \cdot \vdash E_{4.3} : T_{4.2}}\ \text{bind}$$

D2:

$$\frac{\dfrac{}{\cdot; \Theta; \Delta; \cdot; p_2 : T_{3.1} \vdash p_2 : T_{3.1}} \qquad D3}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau_1|\!); p_2 : T_{3.1} \vdash E_{4.2} : T_{4.3}}\ \text{bind}$$

D1:

$$\frac{\dfrac{}{\cdot; \Theta; \Delta; \cdot; p_1 : T_{0.2} \vdash p_1 : T_{0.2}} \qquad D2}{\cdot; \Theta; \Delta; (\!|\Gamma|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau_1|\!); p_1 : T_{0.2}, p_2 : T_{3.1} \vdash E_{4.1} : T_4}\ \text{release}$$

D0:

$$\dfrac{\dfrac{}{\cdot;\Theta;\Delta;\cdot;y:T_{2.1}\vdash y:T_{2.1}} \qquad D1}{\dfrac{\cdot;\Theta;\Delta;(\!|\Gamma|\!);p_1:T_{0.2},y:T_{2.1},p_2:T_{3.1}\vdash E_4:T_4}{\cdot;\Theta;\Delta;(\!|\Gamma|\!);p_1:T_{0.2},y:T_{2.1}\vdash E_3:T_3}\text{lam}}\text{T-subExpE}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{D0}{\cdot;\Theta;\Delta;(\!|\Gamma|\!);p_1:T_{0.2}\vdash E_2:T_2}\text{lam}}{\cdot;\Theta;\Delta;(\!|\Gamma|\!);p_1:T_{0.2}\vdash E_1:T_1}\text{ret}}{\cdot;\Theta;\Delta;(\!|\Gamma|\!);\cdot\vdash E_0:T_{0.1}}\text{lam}$$

- app:

$$\dfrac{\begin{array}{c}\Theta;\Delta;\Gamma_1\vdash_J e_1:([a<I]\tau_1)\multimap\tau_2\rightsquigarrow e_{t1}\\ \Theta,a;\Delta,a<I;\Gamma_2\vdash_K e_2:\tau_1\rightsquigarrow e_{t2}\qquad \Gamma'\sqsupseteq\Gamma_1\oplus\sum_{a<I}\Gamma_2\qquad H\geqslant J+I+\sum_{a<I}K\end{array}}{\Theta;\Delta;\Gamma'\vdash_H e_1\ e_2:\tau_2\rightsquigarrow E_0}\text{app}$$

$E_0 = \lambda p.E_1$

$E_1 = \mathsf{release} - = p \mathsf{\ in\ } E_2$

$E_2 = \mathsf{bind}\, a = \mathsf{store}() \mathsf{\ in\ } E_3$

$E_3 = \mathsf{bind}\, b = e_{t1}\ a \mathsf{\ in\ } E_4$

$E_4 = \mathsf{bind}\, c = \mathsf{store!}() \mathsf{\ in\ } E_5$

$E_5 = \mathsf{bind}\, d = \mathsf{store}() \mathsf{\ in\ } b\ (coerce1\ e_{t2}\ c)\ d$

$T_0 = [H + count(\Gamma')]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{0.11} = [J + I + \sum_{a<I} K + count(\Gamma_1) + count(\sum_{a<I}\Gamma_2)]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{0.1} = [J + I + \sum_{a<I} K + count(\Gamma_1) + count(\sum_{a<I}\Gamma_2)]\,\mathbf{1}$

$T_{0.2} = \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{0.3} = \mathbb{M}(J + I + \sum_{a<I} K + count(\Gamma_1) + count(\sum_{a<I}\Gamma_2))\,(\!|\tau_2|\!)$

$T_1 = [(J + count(\Gamma))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|([a<I]\tau_1) \multimap \tau_2|\!)$

$T_{1.1} = [(J + count(\Gamma))]\,\mathbf{1}$

$T_{1.11} = \mathbb{M}(J + count(\Gamma))\,[(J + count(\Gamma))]\,\mathbf{1}$

$T_{1.12} = \mathbb{M}(I + \sum_{a<I} K + count(\sum_{a<I}\Gamma_2))\,(\!|\tau_2|\!)$

$T_{1.13} = \mathbb{M}(\sum_{a<I} K + count(\sum_{a<I}\Gamma_2))\,T_{1.14}$

$T_{1.131} = \mathbb{M}(\sum_{a<I} K + count(\sum_{a<I}\Gamma_2))\,T_{1.15}$

$T_{1.14} = [(\sum_{a<I} K + count(\sum_{a<I}\Gamma_2))]\,!_{a<I}\mathbf{1} = [\sum_{a<I}(K + count(\Gamma_2))]\,!_{a<I}\mathbf{1}$

$T_{1.15} =!_{a<I}\,[(K + count(\Gamma_2))]\,\mathbf{1}$

$T_{1.2} = \mathbb{M}\,0\,(\!|([a<I]\tau_1) \multimap \tau_2|\!)$

$T_2 = [(J + count(\Gamma))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(!_{a<I}\,\mathbb{M}\,0\,(\!|\tau_1|\!)) \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{2.1} = [(J + count(\Gamma))]\,\mathbf{1}$

$T_{2.2} = \mathbb{M}\,0\,((!_{a<I}\,\mathbb{M}\,0\,(\!|\tau_1|\!)) \multimap [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!))$

$T_{2.21} = (!_{a<I}\,\mathbb{M}\,0\,(\!|\tau_1|\!)) \multimap [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{2.22} = [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_3 = \mathbb{M}\,0\,(\!|\tau_2|\!)$

$T_{3.1} = \mathbb{M}\,I\,(\!|\tau_2|\!)$

$T_4 = \mathbb{M}\,0\,(\!|\tau_1|\!)$

$T_{4.1} =!_{a<I}\,\mathbb{M}\,0\,(\!|\tau_1|\!)$

$T_5 = [(K + count(\Gamma_2))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_1|\!)$

$T_{5.0} =!_{a<I}([(K + count(\Gamma_2))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_1|\!))$

$T_{5.1} =!_{a<I}\,[(K + count(\Gamma_2))]\,\mathbf{1} \multimap !_{a<I}\,\mathbb{M}\,0\,(\!|\tau_1|\!)$

D0.7:

$$\dfrac{}{\cdot; \Theta; \Delta; \cdot; c : T_{1.15} \vdash c : T_{1.15}}\ \text{T-var}$$

D0.6:

$$\dfrac{\dfrac{\dfrac{}{\cdot; \Theta, a; \Delta, a < I; (\!|\Gamma_2|\!); \cdot \vdash e_{t2} : T_5}\ \text{IH}}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); \cdot \vdash !e_{t2} : T_{5.0}}\ \text{subExpI} \qquad D0.7}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); c : T_{1.15} \vdash coerce1\ !e_{t2}\ c : T_{4.1}}\ \text{Lemma 32}$$

D0.5:

$$\dfrac{\dfrac{}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); b : T_{2.21} \vdash b : T_{2.21}} \qquad D0.6}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); b : T_{2.21}, c : T_{1.15} \vdash b\ (coerce1\ !e_{t2}\ c) : T_{2.22}}\ \text{T-app}$$

D0.4:

$$\dfrac{D0.5 \qquad \dfrac{}{\cdot; \Theta; \Delta; \cdot; d : [I]\,\mathbf{1} \vdash d : [I]\,\mathbf{1}}}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); b : T_{2.21}, c : T_{1.15}, d : [I]\,\mathbf{1} \vdash b\ (coerce1\ !e_{t2}\ c)\ d : T_3}$$

D0.3:

$$\dfrac{\dfrac{}{\cdot; \Theta; \Delta; \cdot; \cdot \vdash \mathsf{store}() : \mathbb{M}\,I\,[I]\,\mathbf{1}} \qquad D0.4}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); b : T_{2.21}, c : T_{1.15} \vdash E_5 : T_{3.1}}\ \text{bind}$$

D0.21:

$$\dfrac{}{\cdot; \Theta; \Delta \vdash T_{1.14} <: T_{1.15}}\ \text{sub-bSum}$$

D0.2:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot; \Theta; \Delta; \cdot; \cdot \vdash !() :!_{a<I}\mathbf{1}}}{\cdot; \Theta; \Delta; \cdot; \cdot \vdash \mathsf{store!}() : T_{1.13}} \qquad D0.21}{\cdot; \Theta; \Delta; \cdot; \cdot \vdash \mathsf{store!}() : T_{1.131}}\ \text{T-sub} \qquad D0.3}{\cdot; \Theta; \Delta; \sum_{a<I} (\!|\Gamma_2|\!); b : T_{2.21} \vdash E_4 : T_{1.12}}\ \text{bind}$$

D0.12:

$$\frac{}{\cdot;\Theta;\Delta;.;a:T_{2.1}\vdash a:T_{2.1}}\ \text{T-var}$$

D0.11:

$$\frac{}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!);.\vdash e_{t1}:T_1}\ \text{IH1}$$

D0.1:

$$\frac{\dfrac{D0.11 \qquad D0.12}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!);a:T_{2.1}\vdash e_{t1}\ a:T_{2.2}}\ \text{app} \qquad D0.2}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!)\oplus\sum\limits_{a<I}(\!|\Gamma_2|\!);a:T_{2.1}\vdash E_3:T_{1.12}}\ \text{bind}$$

D0:

$$\frac{\dfrac{}{\cdot;\Theta;\Delta;\cdot;\cdot\vdash \mathsf{store}():T_{1.11}} \qquad D0.1}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!)\oplus\sum\limits_{a<I}(\!|\Gamma_2|\!);\cdot\vdash E_2:T_{0.3}}\ \text{bind}$$

D0.0:

$$\frac{\dfrac{}{\Theta;\Delta\vdash\Gamma'\sqsubseteq\Gamma_1\oplus\sum\limits_{a<I}\Gamma_2}\ \text{By inversion}}{\Theta;\Delta\vdash(\!|\Gamma'|\!)<:(\!|\Gamma_1\oplus\sum\limits_{a<I}\Gamma_2|\!)}\ \text{Lemma 25}$$

Main derivation:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot;\Theta;\Delta;\cdot;p:T_{0.1}\vdash p:T_{0.1}} \qquad D0}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!)\oplus\sum\limits_{a<I}(\!|\Gamma_2|\!);p:T_{0.1}\vdash E_1:T_{0.2}}\ \text{release}}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!)\oplus\sum\limits_{a<I}(\!|\Gamma_2|\!);\cdot\vdash E_0:T_{0.11}}}{\cdot;\Theta;\Delta;(\!|\Gamma_1|\!)\oplus(\!|\sum\limits_{a<I}\Gamma_2|\!);\cdot\vdash E_0:T_{0.11}}\ \text{Lemma 24}}{\cdot;\Theta;\Delta;(\!|\Gamma_1\oplus\sum\limits_{a<I}\Gamma_2|\!);\cdot\vdash E_0:T_{0.11}}\ \text{Lemma 23} \qquad D0.0}{\cdot;\Theta;\Delta;(\!|\Gamma'|\!);\cdot\vdash E_0:T_0}\ \text{T-sub,T-weaken}$$

- fix:

$$\frac{\begin{array}{c}\Theta,b;\Delta,b<L;\Gamma,x:[a<I]\sigma\vdash_K e:\tau\rightsquigarrow e_t \\ \tau[0/a]<:\mu \qquad \Theta,a,b;\Delta,a<I,b<L\vdash\tau[(b+1+\mathop{\triangle}\limits_{b}^{b+1,a} I)/b]<:\sigma \\ \Gamma'\sqsubseteq\sum\limits_{b<L}\Gamma \qquad L,M\geqslant\mathop{\triangle}\limits_{b}^{0,1} I \qquad N\geqslant M-1+\sum\limits_{b<L}K\end{array}}{\Theta;\Delta;\Gamma'\vdash_N \mathsf{fix}x.e:\mu\rightsquigarrow E_0}\ \text{T-fix}$$

$E_0=\mathsf{fix}Y.E_1$

$E_1 = \lambda p.E_2$

$E_2 = \textsf{release} - = p \textsf{ in } E_3$

$E_3 = \textsf{bind}\, A = \textsf{store}() \textsf{ in } E_4$

$E_4 = \textsf{let}\,!\,x = (E_{4.1}\ E_{4.2}) \textsf{ in } E_5$

$E_{4.1} = coerce1\ !Y$

$E_{4.2} = (\lambda u.!())\ A$

$E_5 = \textsf{bind}\, C = \textsf{store}() \textsf{ in } E_6$

$E_6 = e_t\ C$

$$
\begin{array}{l}
cost(b') \triangleq \\
if\ (0 \leqslant b' < (\underset{b}{\overset{0,1}{\bigcirc\!\!\!\!\triangle}} I(b)))\ then \\
\quad K(b') + I(b') + count(\Gamma(b')) + (\sum_{a<I(b')} cost((b' + 1 + \underset{b}{\overset{b'+1,a}{\bigcirc\!\!\!\!\triangle}} I(b)))) \\
else \\
\quad 0
\end{array}
$$

$\tau'(b') = [cost(b')]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau(b')|\!)$

$T_{0.0} = \tau'[(b' + 1 + \underset{b}{\overset{b'+1,a}{\bigcirc\!\!\!\!\triangle}} I)/b']$

$T_0 = [(N + count(\Gamma'))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\mu|\!)$

$T_{0.1} = [(M - 1 + \sum_{b'<L} K) + count(\sum_{b'<L} \Gamma)]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau(0)|\!)$

$b'' = (b' + 1 + \underset{b}{\overset{b'+1,a}{\bigcirc\!\!\!\!\triangle}} I)$

$T_{1.0} = !_{a<I(b')}([cost(b'')]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau(b'')|\!))$

$T_1 = !_{a<I(b')}\,[cost(b'')]\,\mathbf{1} \multimap !_{a<I(b')}\,\mathbb{M}\,0\,(\!|\tau(b'')|\!)$

$T_{1.1} = !_{a<I(b')}\,\mathbb{M}\,0\,(\!|\tau(b'')|\!)$

$T_{1.11} = \mathbb{M}\,0\,(\!|\tau(b'')|\!)$

$T_{1.12} = \mathbb{M}\,0\,(\!|\sigma|\!)$

$T_2 = [\sum_{a<I(b')} cost(b'')]\,\mathbf{1}$

$T_{3.0} = \sum_{a<I} cost(b'')!_{a<I}\mathbf{1}$

$T_3 = !_{a<I}[cost(b'')]\mathbf{1}$

$T_4 = \mathbb{M}(K(b') + I(b') + count(\Gamma(b')))\,(\!|\tau(b')|\!)$

$T_{4.1} = \mathbb{M}(K(b') + I(b') + count(\Gamma(b')))\,[(K(b') + I(b') + count(\Gamma(b')))]\,\mathbf{1}$

$T_{4.2} = [(K(b') + I(b') + count(\Gamma(b')))]\,\mathbf{1}$

$T_5 = [(K(b') + I(b') + count(\Gamma(b')))]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau(b')|\!)$

$T_{c0} = \mathbf{1} \multimap\, !_{a<I}\mathbf{1}$

$T_{c0.1} = [0]\,(\mathbf{1} \multimap\, !_{a<I}\mathbf{1})$

$T_{c1} = [\sum_{a<I} cost(b'')]\,\mathbf{1} \multimap [\sum_{a<I} cost(b'')]\,!_{a<I}\mathbf{1}$

D5.2:

$$\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; C : T_{4.2} \vdash C : T_{4.2}}\ \text{var}$$

D5.10:

$$\dfrac{\dfrac{}{.; \Theta, b'; \Delta, b' < L \vdash \tau(b'') <: \sigma}\ \text{Given}}{.; \Theta, b'; \Delta, b' < L \vdash \llparenthesis \tau(b'') \rrparenthesis <: \llparenthesis \sigma \rrparenthesis}\ \text{Lemma 27}$$

D5.1:

$$\dfrac{\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L; \llparenthesis \Gamma \rrparenthesis, x :_{a<I(b')} T_{1.12}; \cdot \vdash e_t : T_5}\ \text{IH} \qquad D5.10}{\cdot; \Theta, b'; \Delta, b' < L; \llparenthesis \Gamma \rrparenthesis, x :_{a<I(b')} T_{1.11}; \cdot \vdash e_t : T_5}\ \text{T-weaken}$$

D5:

$$\dfrac{D5.1 \qquad D5.2}{\cdot; \Theta, b'; \Delta, b' < L; \llparenthesis \Gamma \rrparenthesis, x :_{a<I(b')} T_{1.11}; C : T_{4.2} \vdash e_t\ C : \mathbb{M}\, 0\, \llparenthesis \tau(b') \rrparenthesis}\ \text{app}$$

D4:

$$\dfrac{\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; \cdot \vdash \mathsf{store}() : T_{4.1}} \qquad D5}{\cdot; \Theta, b'; \Delta, b' < L; \llparenthesis \Gamma \rrparenthesis, x :_{a<I(b')} T_{1.1}; C : T_{4.2} \vdash E_5 : T_4}$$

D3.2:

$$\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L; Y :_{a<I} T_{0.0}; \cdot \vdash !Y : T_{1.0}}\ \text{Lemma 29}$$

D3.11:

$$\dfrac{D3.2}{\cdot; \Theta, b'; \Delta, b' < L; Y :_{a<I} T_{0.0}; \cdot \vdash coerce1\ (!Y) : T_1}\ \text{Lemma 32}$$

D3.12:

$$\dfrac{}{.; \Theta, b'; \Delta, b' < L \vdash T_{3.0} <: T_3}\ \text{sub-bSum}$$

Dc2:

$$\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L \vdash T_{c0.1} <: T_{c1}}\ \text{sub-potArrow}$$

Dc1:

$$\dfrac{\dfrac{\dfrac{}{\cdot; \Theta, b'; \Delta, b' < L, a < I; \cdot; \cdot \vdash () : \mathbf{1}}\ \text{T-unit}}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; u : \mathbf{1} \vdash !() :!_{a<I} \mathbf{1}}\ \text{T-subExpI,T-weaken}}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; \cdot \vdash \lambda u.!() : T_{c0}}\ \text{T-lam}$$

Dc:

$$\dfrac{\dfrac{Dc1 \qquad \dfrac{}{\cdot; \Theta, b'; \Delta, b' < L \vdash T_{c0} <: T_{c0.1}}\ \text{sub-potZero}}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; \cdot \vdash \lambda u.!() : T_{c0.1}}\ \text{T-sub} \qquad Dc2}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; \cdot \vdash \lambda u.!() : T_{c1}}\ \text{T-sub}$$

D3.1:

$$\dfrac{D3.11 \qquad \dfrac{\dfrac{Dc \qquad \dfrac{}{\cdot;\Theta,b';\Delta,b'<L;\cdot;A:T_2\vdash A:T_2}\,\text{var}}{\cdot;\Theta,b';\Delta,b'<L;\cdot;A:T_2\vdash(\lambda u.!())\ A:T_{3.0}}\,\text{T-app} \qquad D3.12}{\cdot;\Theta,b';\Delta,b'<L;\cdot;A:T_2\vdash(\lambda u.!())\ A:T_3}\,\text{T-sub}}{\cdot;\Theta,b';\Delta,b'<L;Y:_{a<I}T_{0.0};A:T_2\vdash E_{4.1}\ E_{4.2}:T_{1.1}}\,\text{app}$$

D3:

$$\dfrac{D3.1 \qquad D4}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma|\!),Y:_{a<I}T_{0.0};A:T_2\vdash E_4:T_4}$$

D2:

$$\dfrac{\dfrac{}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma'|\!);\cdot\vdash\mathsf{store}():\mathbb{M}(\sum\limits_{a<I(b')}cost(b''))T_2} \qquad D3}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma'|\!),Y:_{a<I}T_{0.0};\cdot\vdash E_3:\mathbb{M}(\,cost(b'))(\!|\tau(b')|\!)}$$

D1:

$$\dfrac{\dfrac{}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma|\!);p:[cost(b')]\,\mathbf{1}\vdash p:[cost(b')]\,\mathbf{1}} \qquad D2}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma|\!),Y:_{a<I}T_{0.0};p:[cost(b')]\,\mathbf{1}\vdash E_2:\mathbb{M}(\,0)(\!|\tau(b')|\!)}\,\text{release}$$

D0:

$$\dfrac{\dfrac{\dfrac{\dfrac{D1}{\cdot;\Theta,b';\Delta,b'<L;(\!|\Gamma|\!),Y:_{a<I}T_{0.0};\cdot\vdash E_1:\tau'(b')}\,\text{T-fix}}{\cdot;\Theta;\Delta;\sum\limits_{a<L}(\!|\Gamma|\!);\cdot\vdash E_0:\tau'(0)}\,\text{Claim}}{\cdot;\Theta;\Delta;\sum\limits_{a<L}(\!|\Gamma|\!);\cdot\vdash E_0:T_{0.1}}\,\text{Lemma 24}}{\dfrac{\cdot;\Theta;\Delta;(\!|\sum\limits_{a<L}\Gamma|\!);\cdot\vdash E_0:T_{0.1}}{\cdot;\Theta;\Delta;(\!|\Gamma'|\!);\cdot\vdash E_0:T_{0.1}}\,\text{Lemma 25,T-weaken}}$$

Main derivation:

$$\dfrac{D0}{\cdot;\Theta;\Delta;(\!|\Gamma'|\!);\cdot\vdash E_0:T_0}\,\text{T-sub}$$

Claim:

$\tau'(0)=[(M-1+\sum_{b'<L}K)+count(\sum_{b'<L}\Gamma)]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\!|\tau(0)|\!)$

Proof.

It suffices to prove that

$cost(0)=(M-1+\sum_{b'<L}K)+count(\sum_{b'<L}\Gamma)$

From Definition of *cost* we know that

$$\begin{array}{rcll} cost(0) & = & (\sum_{b'<L}I(b')+\sum_{b'<L}K(b'))+\sum_{b'<L}count(\Gamma) & \\ & = & (M-1+\sum_{b'<L}K(b'))+\sum_{b'<L}count(\Gamma) & \text{Definition of } I \text{ and } M \\ & = & (M-1+\sum_{b'<L}K)+count(\sum_{b'<L}\Gamma) & \text{Lemma 26} \end{array}$$

$\square$

□

Lemma 23 (Relation b/w dlPCF context and its translation - binary sum)**.** $\forall \Gamma_1, \Gamma_2 \in dlPCF$. $(\!|\Gamma_1 \oplus \Gamma_2|\!) = (\!|\Gamma_1|\!) \oplus (\!|\Gamma_2|\!)$

Proof. Proof by induction on Γ_1

$\underline{\Gamma_1 = .}$

$$\begin{array}{lcll} (\!|. \oplus \Gamma_2|\!) & = & (\!|\Gamma_2|\!) & \text{Definition 3} \\ & = & (\!|.|\!) + (\!|\Gamma_2|\!) & \text{Definition 4} \end{array}$$

$\underline{\Gamma_1 = \Gamma_1', x : [-]-}$

When $x : [-]- \notin \Gamma_2$

$$\begin{array}{lcll} (\!|\Gamma_1', x : [a < I]\tau \oplus \Gamma_2|\!) & = & (\!|(\Gamma_1' \oplus \Gamma_2), x : [a < I]\tau|\!) & \text{Definition 3} \\ & = & (\!|(\Gamma_1' \oplus \Gamma_2)|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau|\!) & \text{Definition 21} \\ & = & ((\!|\Gamma_1'|\!) \oplus (\!|\Gamma_2|\!)), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau|\!) & \text{IH} \\ & = & (\!|\Gamma_1'|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau|\!) \oplus (\!|\Gamma_2|\!) & \text{Definition 4} \\ & = & (\!|\Gamma_1', x : [a < I]\tau|\!) \oplus (\!|\Gamma_2|\!) & \text{Definition 4} \end{array}$$

When $x : [b < J]\tau[I + b/c] \in \Gamma_2$

Let $(\!|\Gamma_1', x : [a < I]\tau[a/c] \oplus \Gamma_2', x : [b < J]\tau[I + b/c]|\!) = \Gamma_r$

$$\begin{array}{lcll} \Gamma_r & = & (\!|(\Gamma_1' \oplus \Gamma_2'), x : [c < (I + J)]\tau|\!) & \text{Definition 3} \\ & = & (\!|(\Gamma_1' \oplus \Gamma_2')|\!), x :_{c<(I+J)} \mathbb{M}\, 0\, (\!|\tau|\!) & \text{Definition 21} \\ & = & ((\!|\Gamma_1'|\!) \oplus (\!|\Gamma_2'|\!)), x :_{c<(I+J)} \mathbb{M}\, 0\, (\!|\tau|\!) & \text{IH} \\ & = & (\!|\Gamma_1'|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau|\!)[a/c] \oplus (\!|\Gamma_2'|\!), x :_{b<J} \mathbb{M}\, 0\, (\!|\tau|\!)[I + b/c] & \text{Definition 4} \\ & = & (\!|\Gamma_1'|\!), x :_{a<I} \mathbb{M}\, 0\, (\!|\tau[a/c]|\!) \oplus (\!|\Gamma_2'|\!), x :_{b<J} \mathbb{M}\, 0\, (\!|\tau[I + b/c]|\!) & \text{Lemma 28} \\ & = & (\!|\Gamma_1', x : [a < I]\tau[a/c]|\!) \oplus (\!|\Gamma_2', x : [b < J]\tau[I + b/c]|\!) & \text{Definition 4} \end{array}$$

□

Lemma 24 (Relation b/w dlPCF context and its translation - bounded sum)**.** $\forall \Gamma \in dlPCF$. $(\!|\sum_{a<I} \Gamma|\!) = \sum_{a<I} (\!|\Gamma|\!)$

Proof. Proof by induction on Γ

$\underline{\Gamma = .}$

$$\begin{array}{lcll} (\!|\sum_{a<I} .|\!) & = & (\!|.|\!) & \text{Definition 1} \\ & = & . & \text{Definition 21} \\ & = & \sum_{a<I} (\!|.|\!) & \text{Definition 2} \end{array}$$

$\underline{\Gamma = \Gamma', x : [-]-}$

Let $(\!|\sum_{a<I}(\Gamma', x : [b < J]\sigma[\sum_{d<a} J[d/a] + b/c])|\!) = \Gamma_r$

$$\begin{array}{lcll} \Gamma_r & = & (\!|\sum_{a<I}(\Gamma'), x : [c < \sum_{a<I} J]\sigma|\!) & \text{Definition 1} \\ & = & (\!|\sum_{a<I}(\Gamma')|\!), x :_{c<\sum_{a<I} J} \mathbb{M}\, 0\, (\!|\sigma|\!) & \text{Definition 21} \\ & = & \sum_{a<I} (\!|\Gamma'|\!), x :_{c<\sum_{a<I} J} \mathbb{M}\, 0\, (\!|\sigma|\!) & \text{IH} \\ & = & \sum_{a<I} ((\!|\Gamma'|\!), x :_{b<J} \mathbb{M}\, 0\, (\!|\sigma|\!)[\sum_{d<a} J[d/a] + b/c]) & \text{Definition 2} \\ & = & \sum_{a<I} ((\!|\Gamma'|\!), x :_{b<J} \mathbb{M}\, 0\, (\!|\sigma[\sum_{d<a} J[d/a] + b/c]|\!)) & \text{Lemma 28} \\ & = & \sum_{a<I} (\!|\Gamma', x : [b < J]\sigma[\sum_{d<a} J[d/a] + b/c]|\!) & \text{Definition 21} \end{array}$$

□

Lemma 25 (Relation b/w dlPCF context and its translation - subtyping)**.** $\forall \Gamma, \Gamma' \in dlPCF$. $\Theta; \Delta \models \Gamma_1 \sqsubseteq \Gamma_2 \implies .; \Theta; \Delta \models (\!|\Gamma_1|\!) <: (\!|\Gamma_2|\!)$

Proof. Proof by induction on the $\Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2$ relation

1. dlpcf-sub-mBase:

$$\frac{}{.;\Theta;\Delta \vdash (\!|\Gamma_1|\!) <: .}\text{ sub-mBase}$$

2. dlpcf-sub-mInd:

D4:

$$\frac{\frac{}{.;\Theta;\Delta \vdash \Gamma_1/x <: \Gamma_2}\text{ By inversion}}{.;\Theta;\Delta \vdash (\!|\Gamma_1|\!)/x <: (\!|\Gamma_2|\!)}\text{ IH}$$

D3:

$$\frac{}{\Theta;\Delta \vdash I \leqslant J}\text{ By inversion}$$

D2:

$$\frac{\frac{\frac{}{.;\Theta,a;\Delta,a<I \vdash \tau' <: \tau}\text{ By inversion}}{.;\Theta,a;\Delta,a<I \vdash (\!|\tau'|\!) <: (\!|\tau|\!)}\text{ Lemma 27}}{.;\Theta,a;\Delta,a<I \vdash \mathbb{M}\,0\,(\!|\tau'|\!) <: \mathbb{M}\,0\,(\!|\tau|\!)}$$

D1:

$$\frac{\frac{}{x : [a<J]\tau' \in \Gamma_1}\text{ By inversion}}{x :_{a<J} \mathbb{M}\,0\,(\!|\tau'|\!) \in (\!|\Gamma_1|\!)}\text{ Definition 21}$$

Main derivation:

$$\frac{\frac{D1 \quad D2 \quad D3 \quad D4}{.;\Theta;\Delta \vdash (\!|\Gamma_1|\!) <: (\!|\Gamma_2'|\!), x :_{a<I} \mathbb{M}\,0\,(\!|\tau|\!)}}{.;\Theta;\Delta \vdash (\!|\Gamma_1|\!) <: (\!|\Gamma_2', x : [a<I]\tau|\!)}$$

□

Lemma 26. $\forall L, \Gamma.$
$\sum_{a<L} count(\Gamma) = count(\sum_{a<L} \Gamma)$

Proof. By induction on Γ

$\underline{\Gamma = .}$
From Definition of *count* we know that $count(.) = 0$ therefore
$\sum_{a<L} count(.) = 0$

From Definition 2 we know that $\sum_{a<L} . = .$
Therefore again from Definition of *count* we know that $count(.) = 0$
And we are done

$\underline{\Gamma = \Gamma', x :_{b<J} \tau}$

$count(\sum_{a<L} \Gamma', x :_{b<J} \tau)$	$=$	$count(\sum_{a<L} \Gamma', x :_{c<\sum_{a<L} J} \sigma)$	Definition 2
		where $\tau = \sigma[(\sum_{d<a} J[d/a] + b)/c]$	
	$=$	$count(\sum_{a<L} \Gamma') + \sum_{a<L} J$	Definition $count(.)$
	$=$	$\sum_{a<L} count(\Gamma') + \sum_{a<L} J$	IH
	$=$	$\sum_{a<L} count(\Gamma', x :_{b<J} \tau)$	

□

Lemma 27 (Subtyping is preserved by translation)**.** $\Theta; \Delta \vdash^D \sigma <: \tau \implies \Theta; \Delta \vdash^A (\!|\sigma|\!) <: (\!|\tau|\!)$

Proof. By induction on $\Theta; \Delta \vdash^D \sigma <: \tau$

1. $[a < I]\sigma_1 \multimap \sigma_2 <: [a < J]\tau_1 \multimap \tau_2$:

   D1:

$$\dfrac{\dfrac{\dfrac{}{\Theta; \Delta \vdash^A I \leqslant J}\text{ By inversion}}{\Theta; \Delta \vdash^A [J]\,\mathbf{1} <: [I]\,\mathbf{1}} \qquad \dfrac{\dfrac{}{\Theta; \Delta \vdash^A (\!|\sigma_2|\!) <: (\!|\tau_2|\!)}\text{ IH2}}{\Theta; \Delta \vdash^A \mathbb{M}\,0\,(\!|\sigma_2|\!) <: \mathbb{M}\,0\,(\!|\tau_2|\!)}}{\Theta; \Delta \vdash^A [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\sigma_2|\!) <: [J]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)}$$

   Main derivation:

$$\dfrac{\dfrac{\dfrac{}{\Theta, a; \Delta \vdash^A I \leqslant J}\text{ By inversion} \qquad \dfrac{\dfrac{}{\Theta; \Delta \vdash^A (\!|\tau_1|\!) <: (\!|\sigma_1|\!)}\text{ IH1}}{\Theta; \Delta \vdash^A \mathbb{M}\,0\,(\!|\tau_1|\!) <: \mathbb{M}\,0\,(\!|\sigma_1|\!)}}{\Theta; \Delta \vdash^A !_{a<J}\,\mathbb{M}\,0\,(\!|\tau_1|\!) <: !_{a<I}\,\mathbb{M}\,0\,(\!|\sigma_1|\!)} \qquad D1}{\Theta; \Delta \vdash^A !_{a<I}\,\mathbb{M}\,0\,(\!|\sigma_1|\!) \multimap [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\sigma_2|\!) <: !_{a<J}\,\mathbb{M}\,0\,(\!|\tau_1|\!) \multimap [J]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)}$$

□

Lemma 28 (Index Substitution lemma)**.** $\forall \tau \in dlPCF, J.$
$(\!|\tau|\!)[J/b] = (\!|\tau[J/b]|\!)$

Proof. By induction on τ

1. $\tau = b$:

$$\begin{array}{ll} & (\!|b|\!)[J/b] \\ = & b \\ = & (\!|b[J/b]|\!) \end{array}$$

2. $\tau = [a < I]\tau_1 \multimap \tau_2$:

$$\begin{array}{lll} & (\!|[a < I]\tau_1 \multimap \tau_2|\!)[J/b] & \\ = & !_{a<I}\,\mathbb{M}\,0(\!|\tau_1|\!) \multimap [I]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)[J/b] & \\ = & !_{a<I[J/b]}\,\mathbb{M}\,0(\!|\tau_1|\!)[J/b] \multimap [I][J/b]\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2|\!)[J/b] & \\ = & !_{a<I[J/b]}\,\mathbb{M}\,0(\!|\tau_1[J/b]|\!) \multimap [I][J/b]\mathbf{1} \multimap \mathbb{M}\,0\,(\!|\tau_2[J/b]|\!) & \text{(From IH)} \\ = & (\!|[a < I[J/b]]\tau_1[J/b] \multimap \tau_2[J/b]|\!) & \end{array}$$

□

Lemma 29. $\Psi; \Theta; \Delta; x :_{a<I} \tau; \cdot \vdash !x :!_{a<I}\tau$

Proof.

$$\dfrac{\dfrac{\dfrac{}{\Psi; \Theta, a; \Delta, a < I; x :_{b<1} \tau[a + b/a]; \cdot \vdash x : \tau}\text{ T-var2}}{\Psi; \Theta; \Delta; \sum_{a<I} x :_{b<1} \tau[a + b/a]; \cdot \vdash !x :!_{a<I}\tau}\text{ T-subExpI}}{\Psi; \Theta; \Delta; x :_{a<I} \tau; \cdot \vdash !x :!_{a<I}\tau}\text{ Lemma 30}$$

□

Lemma 30. $\sum_{a<I} x :_{b<1} \tau[a + b/a] = x_{a<I}\tau$

Proof. It suffices to prove that
$\sum_{a<I} x :_{b<1} \tau[a+b/a] = x_{c<I}\tau[c/a]$

From Definition 2 it suffices to prove that
$\sum_{a<I} x :_{b<1} \tau[a+b/a] = x_{c<\sum_{a<I} 1}\tau[c/a]$

Again from Definition 2 it suffices to prove that
$\tau[c/a][(\sum_{d<a} 1[d/a]+b)/c] = \tau[a+b/a]$

$\tau[c/a][(\sum_{d<a} 1[d/a]+b)/c] =$
$\tau[c/a][(\sum_{d<a} 1[d/a]+b)/c] =$
$\tau[c/a][(a+b)/c] =$
$\tau[(a+b)/a]$

So, we a re done $\square$

Definition 31 (Coercion function). $coerce1\ F\ X \triangleq$
$\mathsf{let}\,!\,f = F\ \mathsf{in}\ \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x)$

Lemma 32 (Coerce is well-typed). $\cdot;\cdot;\cdot;\cdot;\cdot \vdash coerce1 :!_{a<I}(\tau_1 \multimap \tau_2) \multimap !_{a<I}\tau_1 \multimap !_{a<I}\tau_2$

Proof. D2.2

$$\frac{}{\cdot;a;a<I;x :_{b<1} \tau_1[a+b/a];\cdot \vdash x : \tau_1}$$

D2.1:

$$\frac{}{\cdot;a;a<I;f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a];\cdot \vdash f : \tau_1 \multimap \tau_2}$$

D2:

$$\dfrac{\dfrac{\dfrac{D2.1 \qquad D2.2}{\cdot;a;a<I;f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a], x :_{b<1} \tau_1[a+b/a];\cdot \vdash (f\ x) : \tau_2}}{\cdot;\cdot;\cdot;\sum\limits_{a<I} f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a], x :_{b<1} \tau_1[a+b/a];\cdot \vdash !(f\ x) :!_{a<I}\tau_2}\ \text{T-subExpI}}{\cdot;\cdot;\cdot;f :_{a<I} (\tau_1 \multimap \tau_2), x :_{a<I} \tau_1;\cdot \vdash !(f\ x) :!_{a<I}\tau_2}\ \text{Lemma 33}$$

D1:

$$\frac{\dfrac{}{\cdot;\cdot;\cdot;f :_{a<I} (\tau_1 \multimap \tau_2); X :!_{a<I}\tau_1 \vdash !(f\ x)} \qquad D2}{\cdot;\cdot;\cdot;f :_{a<I} (\tau_1 \multimap \tau_2);\cdot \vdash \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x)}$$

D0:

$$\frac{\dfrac{}{\cdot;\cdot;\cdot;\cdot;F :!_{a<I}(\tau_1 \multimap \tau_2) \vdash F :!_{a<I}(\tau_1 \multimap \tau_2)}\ \text{T-var1} \qquad D1}{\cdot;\cdot;\cdot;\cdot;F :!_{a<I}(\tau_1 \multimap \tau_2) \vdash \mathsf{let}\,!\,f = F\ \mathsf{in}\ \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x)}$$

Main derivation:

$$\frac{\dfrac{D0}{\cdot;\cdot;\cdot;\cdot;F :!_{a<I}(\tau_1 \multimap \tau_2) \vdash \lambda X.\,\mathsf{let}\,!\,f = F\ \mathsf{in}\ \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x)}}{\cdot;\cdot;\cdot;\cdot;\cdot \vdash \lambda F.\lambda X.\,\mathsf{let}\,!\,f = F\ \mathsf{in}\ \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x) :!_{a<I}(\tau_1 \multimap \tau_2) \multimap !_{a<I}\tau_1 \multimap !_{a<I}\tau_2}$$

$\square$

Lemma 33. $\sum_{a<I} f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a], x :_{b<1} \tau_1[a+b/a] = f :_{a<I} \tau_1 \multimap \tau_2, x :_{a<I} \tau_1$

Proof. It suffices to prove that
$\sum_{a<I} f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a], x :_{b<1} \tau_1[a+b/a] = f :_{c<I} (\tau_1 \multimap \tau_2)[c/a], x :_{c<I} \tau_1[c/a]$

From Definition 2 it suffices to prove that
$\sum_{a<I} f :_{b<1} (\tau_1 \multimap \tau_2)[a+b/a], x :_{b<1} \tau_1[a+b/a] = f :_{c<\sum_{a<I} 1} (\tau_1 \multimap \tau_2)[c/a], x :_{c<\sum_{a<I} 1} \tau_1[c/a]$

Again from Definition 2 it suffices to prove that

1. $(\tau_1 \multimap \tau_2)[c/a][(\sum_{d<a} 1[d/a] + b)/c] = (\tau_1 \multimap \tau_2)[a+b/a]$:
   $(\tau_1 \multimap \tau_2)[c/a][(\sum_{d<a} 1[d/a] + b)/c] =$
   $(\tau_1 \multimap \tau_2)[c/a][(\sum_{d<a} 1[d/a] + b)/c] =$
   $(\tau_1 \multimap \tau_2)[c/a][(a+b)/c] =$
   $(\tau_1 \multimap \tau_2)[(a+b)/a]$

2. $\tau_1[c/a][(\sum_{d<a} 1[d/a] + b)/c] = \tau_1[a+b/a]$:
   $\tau_1[c/a][(\sum_{d<a} 1[d/a] + b)/c] =$
   $\tau_1[c/a][(\sum_{d<a} 1[d/a] + b)/c] =$
   $\tau_1[c/a][(a+b)/c] =$
   $\tau_1[(a+b)/a]$

So, we are done $\square$

1.5.2 Cross-language model: dlPCF to λ-amor

Definition 34 (Logical relation for dlPCF to λ-Amor)**.**

$$
\begin{array}{lcl}
\lfloor \mathsf{b} \rfloor_V & \triangleq & \{({}^s v, {}^t v) \mid {}^s v \in [\![\mathsf{b}]\!] \wedge {}^t v \in [\![\mathsf{b}]\!] \wedge {}^s v = {}^t v\} \\
\lfloor [a<I]\tau_1 \multimap \tau_2 \rfloor_V & \triangleq & \{(\lambda x.e_s, \lambda x.\lambda p.\,\mathsf{let}\,!\,x = y \;\mathsf{in}\; e_t) \mid \forall e'_s, e'_t. \\
 & & (e'_s, e'_t) \in \lfloor [a<I]\tau_1 \rfloor_{NE} \implies (e_s[e'_s/x], e_t[e'_t/y][()/p]) \in \lfloor \tau_2 \rfloor_E\} \\
\\
\lfloor \tau \rfloor_E & \triangleq & \{(e_s, e_t) \mid \forall^s v. e_s \Downarrow {}^s v \implies \exists^t v_t, {}^t v_f, J. e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau \rfloor_V\} \\
\\
\lfloor [a<I]\tau \rfloor_{NE} & \triangleq & \{(e_s, e_t) \mid \exists e'_t. e_t = \mathit{coerce}1\; !e'_t\; !() \wedge \forall 0 \leqslant i < I. (e_s, e'_t()) \in \lfloor \tau[i/a] \rfloor_E\}
\end{array}
$$

Definition 35 (Interpretation of typing contexts)**.**

$$
\begin{array}{lcl}
\lfloor \Gamma \rfloor_E & = & \{(\delta_s, \delta_t) \mid \\
 & & (\forall x : [a<J]\tau \in \mathit{dom}(\Gamma). \forall 0 \leqslant j < J. (\delta_s(x), \delta_t(x)) \in \lfloor \tau[j/a] \rfloor_E)
\end{array}
$$

Theorem 36 (Fundamental theorem)**.** $\forall \Theta, \Delta, \Gamma, \tau, e_s, e_t, I, \delta_s, \delta_t.$
$\Theta; \Delta; \Gamma \vdash_I e_s : \tau \rightsquigarrow e_t \wedge (\delta_s, \delta_t) \in \lfloor \Gamma\ \iota \rfloor_E \wedge .\models \Delta\ \iota$
$\implies$
$(e_s \delta_s, e_t\ ()\ \delta_t) \in \lfloor \tau\ \iota \rfloor_E$

Proof. Proof by induction on the translation relation:

1. var:

$$\frac{\Theta; \Delta \models I \geqslant 1 \qquad \Theta; \Delta \vdash \tau'[0/a] <: \tau \qquad \Theta; \Delta \models [a < I]\tau' \Downarrow \qquad \begin{array}{c}\Theta; \Delta \models J \geqslant 0 \\ \Theta; \Delta \models \Gamma \Downarrow\end{array}}{\Theta; \Delta; \Gamma, x : [a < I]\tau' \vdash_J x : \tau \rightsquigarrow \lambda p.\mathsf{release} - = p \mathsf{~in~} \mathsf{bind} - = \uparrow^1 \mathsf{~in~} x} \text{var}$$

$E_1 = \lambda p.\mathsf{release} - = p \mathsf{~in~bind} - = \uparrow^1 \mathsf{~in~} x$

Given: $(\delta_s, \delta_t) \in \lfloor \Gamma, x \rfloor_E$

To prove: $(x\delta_s, E_1\ ()\delta_t) \in \lfloor \tau \rfloor_E$

This means from Definition 34 we need to prove that

$\forall^s v.x\delta_s \Downarrow {}^s v \implies \exists^t v_t, {}^t v_f, J'.E_1\ () \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau \rfloor_V$

This means that given some ${}^s v$ s.t $x\delta_s \Downarrow {}^s v$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J'.E_1\ () \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau \rfloor_V$ (F-DA-V0)

Since we are given that $(\delta_s, \delta_t) \in \lfloor \Gamma, x \rfloor_E$ therefore from Definition 35 we know that

$\forall y : [a < J]\tau'' \in dom(\Gamma, x).\ \forall 0 \leqslant i < J.(\delta_s(y), \delta_t(y)) \in \lfloor \tau''[i/a] \rfloor_E$

This means we also have $(\delta_s(x), \delta_t(x)) \in \lfloor \tau'[0/a] \rfloor_E$. This further means that from Definition 34 we have

$\forall^s v''.\delta_s(x) \Downarrow {}^s v'' \implies \exists J'', {}^t v_t'', {}^t v_f''.\delta_t(x) \Downarrow {}^t v_t'' \Downarrow^{J''} {}^t v_f'' \wedge ({}^s v'', {}^t v_f'') \in \lfloor \tau'[0/a] \rfloor_V$ (F-DA-V1)

We instantiate (F-DA-V1) with ${}^s v$ and in order to prove (F-DA-V0) we choose J' as J'', ${}^t v_t$ as ${}^t v_t''$ and ${}^t v_f$ as ${}^t v_f''$ and we get the desired from (F-DA-V1) and Lemma 37.

2. lam:

$$\frac{\Theta; \Delta; \Gamma, x : [a < I]\tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\begin{array}{c}\Theta; \Delta; \Gamma \vdash_J \lambda x.e : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow \\ \lambda p_1.\mathsf{~ret~} \lambda y.\lambda p_2.\mathsf{~let\,!\,} x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind~} a = \mathsf{store}() \mathsf{~in~} e_t\ a\end{array}} \text{lam}$$

$E_1 = \lambda p_1.\mathsf{~ret~} \lambda y.\lambda p_2.\mathsf{~let\,!\,} x = y \mathsf{~in~release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind~} a = \mathsf{store}() \mathsf{~in~} e_t\ a$

$E_2 = \lambda y.\lambda p_2.\mathsf{~let\,!\,} x = y \mathsf{~in~} E_3$

$E_3 = \mathsf{release} - = p_1 \mathsf{~in~release} - = p_2 \mathsf{~in~bind~} a = \mathsf{store}() \mathsf{~in~} e_t\ a$

Given: $(\delta_s, \delta_t) \in \lfloor \Gamma \rfloor_E$

To prove: $(\lambda x.e\delta_s, E_1()\delta_t) \in \lfloor ([a < I].\tau_1) \multimap \tau_2 \rfloor_E$

This means from Definition 34 we need to prove that

$\forall^s v.\lambda x.e\delta_s \Downarrow {}^s v \implies \exists J', {}^t v_t, {}^t v_f.E_1() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor ([a < I].\tau_1) \multimap \tau_2 \rfloor_V$

This means that given some ${}^s v$ s.t $\lambda x.e\delta_s \Downarrow {}^s v$ it suffices to prove that

$\exists J', {}^t v_t, {}^t v_f . E_1() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor([a < I].\tau_1) \multimap \tau_2\rfloor_V$ (F-DA-L0)

We know that ${}^s v = \lambda x.e\delta_s$. Also from E-app, E-ret we know that ${}^t v_f = E_2$ and $J' = 0$

Therefore it suffices to show $(\lambda x.e\ \delta_s, E_2) \in \lfloor((([a < I].\tau_1) \multimap \tau_2)\iota\rfloor_V$

From Definition 34 it further suffices to prove that
$\forall e'_s, e'_t.(e'_s, e'_t) \in \lfloor[a < I]\tau_1\iota\rfloor_{NE} \implies (e_s[e'_s/x], E_3[e'_t/y][()/p_2]) \in \lfloor\tau_2\ \iota\rfloor_E$ (F-DA-L1)

This means given some e'_s, e'_t s.t $(e'_s, e'_t) \in \lfloor[a < I]\tau_1\iota\rfloor_{NE}$. We need to prove that
$(e_s[e'_s/x], E_2[e'_t/x][()/p_2]) \in \lfloor\tau_2\ \iota\rfloor_E$ (F-DA-L1.1)

Since $(e'_s, e'_t) \in \lfloor[a < I]\tau_1\iota\rfloor_{NE}$ therefore from Definition 34 we have
$\exists e''_t.e'_t = coerce1\ !e''_t\ !() \wedge \forall 0 \leqslant i < I.(e'_s, e''_t()) \in \lfloor\tau_1[i/a]\ \iota\rfloor_E$

Let
$\delta'_s = \delta_s \cup \{x \mapsto e'_s\}$ and
$\delta'_t = \delta_t \cup \{x \mapsto e''_t\ ()\}$

From Definition 35 we know that
$(\delta'_s, \delta'_t) \in \lfloor\Gamma, x : [a < I]\tau_1\ \iota\rfloor_E$

Therefore from IH we have
$(e_s\ \delta'_s, e_t()\ \delta'_t) \in \lfloor\tau_2\ \iota\rfloor_E$ (F-DA-L2)

This means from Definition 34 we have
$\forall {}^s v_b.e_s\ \delta'_s \Downarrow {}^s v_b \implies \exists J_b, {}^t v_{t1}, {}^t v_b.e_t()\ \delta'_t \Downarrow {}^t v_{t1} \Downarrow^{J_b} {}^t v_b \wedge ({}^s v_b, {}^t v_b) \in \lfloor\tau_2\ \iota\rfloor_V$ (F-DA-L3)

Applying Definition 34 on (F-DA-L1.1) we need to prove
$\forall {}^s v_f.e_s[e'_s/x]\delta_s \Downarrow {}^s v_f \implies \exists J_1, {}^t v_t, {}^t v_f.E_2[e'_t/x][()/p_2]\delta_t \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in \lfloor\tau_2\iota\rfloor_V$

This means given some ${}^s v_f$ s.t $e_s[e'_s/x]\delta_s \Downarrow {}^s v_f$ it suffices to prove
$\exists J_1, {}^t v_t, {}^t v_f.E_2[e'_t/x][()/p_2]\delta_t \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in \lfloor\tau_2\iota\rfloor_V$ (F-DA-L4)

Therefore instantiating (F-DA-L3) with ${}^s v_f$ and we get the desired

3. app:

$$\frac{\begin{array}{c}\Theta; \Delta; \Gamma \vdash_J e_1 : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow e_{t1} \\ \Theta, a; \Delta, a < I; \Delta \vdash_K e_2 : \tau_1 \rightsquigarrow e_{t2} \qquad \Gamma' \sqsubseteq \Gamma \oplus \sum_{a<I} \Delta \qquad H \geqslant J + I + \sum_{a<I} K\end{array}}{\Theta; \Delta; \Gamma' \vdash_H e_1\ e_2 : \tau_2 \rightsquigarrow E_1} \text{ app}$$

$E_1 = \lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ } E'_1$
$E'_1 = \mathsf{bind\ } d = \mathsf{store}() \mathsf{\ in\ } b\ (coerce1\ !e_{t2}\ c)\ d$
Given: $(\delta_s, \delta_t) \in \lfloor\Gamma'\ \iota\rfloor_E$

To prove: $(e_1\ e_2\delta_s, E_1()\delta_t) \in \lfloor \tau_2\ \iota \rfloor_E$

This means from Definition 34 we need to prove that
$\forall^s v_f.(e_1\ e_2)\delta_s \Downarrow {}^s v_f \implies \exists J', {}^t v_t, {}^t v_f.E_1() \Downarrow {}^t v_t \Downarrow^{J'}\ {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in \lfloor \tau_2\ \iota \rfloor_V$

This means that given some ${}^s v_f$ s.t $(e_1\ e_2)\delta_s \Downarrow {}^s v_f$ it suffices to prove that
$\exists J', {}^t v_t, {}^t v_f.E_1() \Downarrow {}^t v_t \Downarrow^{J'}\ {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in \lfloor \tau_2\ \iota \rfloor_V$ (F-DA-A0)

<u>IH1</u>
$(e_1\delta_s, e_{t1}()\delta_t) \in \lfloor ([a < I]\tau_1 \multimap \tau_2)\ \iota \rfloor_E$

This means from Definition 34 we have
$\forall^s v_1.e_1\delta_s \Downarrow {}^s v_1 \implies \exists J_1, {}^t v_1', {}^t v_1.e_{t1}()\delta_t \Downarrow {}^t v_1' \Downarrow^{J_1}\ {}^t v_1 \wedge ({}^s v_1, {}^t v_1) \in \lfloor ([a < I]\tau_1 \multimap \tau_2)\ \iota \rfloor_V$

Since we know that $(e_1\ e_2)\delta_s \Downarrow^n {}^s v_f$ therefore we know that $\exists^s v_1$ s.t $e_1\delta_s \Downarrow {}^s v_1$. Therefore we have
$\exists J_1, {}^t v_1', {}^t v_1.e_{t1}()\delta_t \Downarrow {}^t v_1' \Downarrow^{J_1}\ {}^t v_1 \wedge ({}^s v_1, {}^t v_1) \in \lfloor ([a < I]\tau_1 \multimap \tau_2)\ \iota \rfloor_V$ (F-DA-A1)

Since we know that $({}^s v_1, {}^t v_1) \in \lfloor ([a < I]\tau_1 \multimap \tau_2)\ \iota \rfloor_V$

Let ${}^s v_1 = \lambda x.e_{bs}$ and ${}^t v_1 = \lambda x.\lambda p.\ \mathsf{let}\,!\,x = y\ \mathsf{in}\ e_{bt}$
Therefore from Definition 34 we have
$\forall e_s', e_t'.(e_s', e_t') \in \lfloor [a < I]\tau_1\ \iota \rfloor_{NE} \implies (e_{bs}[e_s'/x], e_{bt}[e_t'/x][()/p]) \in \lfloor \tau_2\ \iota \rfloor_E$ (F-DA-A2)

<u>IH2</u>
$(e_2\delta_s, e_{t2}()\delta_t) \in \lfloor \tau_1\ \iota \cup \{a \mapsto 0\} \rfloor_E$
$(e_2\delta_s, e_{t2}()\delta_t) \in \lfloor \tau_1\ \iota \cup \{a \mapsto 1\} \rfloor_E$
...
$(e_2\delta_s, e_{t2}()\delta_t) \in \lfloor \tau_1\ \iota \cup \{a \mapsto I - 1\} \rfloor_E$ (F-DA-A3)

We claim that
$(e_2\delta_s, coerce\ !e_{t2}\ !()\delta_t) \in \lfloor [a < I]\tau_1\ \iota \rfloor_{NE}$

From Definition 31 we know that
$coerce\ F\ X \triangleq$
$\mathsf{let}\,!\,f = F\ \mathsf{in}\ \mathsf{let}\,!\,x = X\ \mathsf{in}!(f\ x)$

therefore the desired holds from Definition 34 and (F-DA-A3)

Instantiating (F-DA-A2) with $e_2\delta_s, coerce\ !e_{t2}\ !()\delta_t$ we get
$(e_{bs}[e_2\delta_s/x], e_{bt}[coerce\ !e_{t2}\ !()\delta_t/x]()) \in \lfloor \tau_2\ \iota \rfloor_E$ (F-DA-A4)

This further means that from Definition 34 we have
$\forall^s v_{bf}.e_{bs}[e_2\delta_s/x] \Downarrow {}^s v_{bf} \implies \exists J_2, {}^t v_{tb}, {}^t v_{bf}.e_{bt}[coerce\ !e_{t2}\ !()\delta_t/x]() \Downarrow {}^t v_{tb} \Downarrow^{J_2}\ {}^t v_{bf} \wedge ({}^s v_{bf}, {}^t v_{bf}) \in \lfloor \tau_2\ \iota \rfloor_V$

Since we know that $(e_1\ e_2)\delta_s \Downarrow^n {}^s v_f$ therefore we know that $\exists^s v_{bf}, n_2$ s.t $e_{bs}[e_2\delta_s/x] \Downarrow^{n_2} {}^s v_{bf}$. Therefore we have

$$\exists J_2, {}^t v_{tb}, {}^t v_{bf}. e_{bt}[coerce\ !e_{t2}\ !()\delta_t/x]() \Downarrow {}^t v_{tb} \Downarrow^{J_2} {}^t v_{bf} \wedge ({}^s v_{bf}, {}^t v_{bf}) \in \lfloor \tau_2\ \iota \rfloor_V \qquad \text{(F-DA-A5)}$$

In order to prove (F-DA-A0) we choose J' as $J_1 + J_2$, ${}^t v_t$ as ${}^t v_{tb}$ and ${}^t v_f$ as ${}^t v_{bf}$, we get the desired from (F-DA-A1) and (F-DA-A5)

4. fix:

$$\frac{\begin{array}{c}\Theta, b; \Delta, b < L; \Gamma, x : [a < I]\sigma \vdash_K e : \tau \rightsquigarrow e_t \\ \tau[0/a] <: \mu \qquad \Theta, a, b; \Delta, a < I, b < L; \Gamma \vdash \tau[(b + 1 + \bigtriangleup_b^{b+1,a} I)/b] <: \sigma \\ \Gamma' \sqsubseteq \sum_{b<L} \Gamma \qquad L, M \geqslant \bigtriangleup_b^{0,1} I \qquad N \geqslant M - 1 + \sum_{b<L} K\end{array}}{\Theta; \Delta; \Gamma' \vdash_N \mathsf{fix} x.e : \mu \rightsquigarrow E_0} \text{ T-fix}$$

$E_0 = \mathsf{fix} Y.E_1$

$E_1 = \lambda p.E_2$

$E_2 = \mathsf{release} - = p \mathsf{\ in\ } E_3$

$E_3 = \mathsf{bind}\ A = \mathsf{store}() \mathsf{\ in\ } E_4$

$E_4 = \mathsf{let}\,!\,x = (E_{4.1}\ E_{4.2}) \mathsf{\ in\ } E_5$

$E_{4.1} = coerce1\ !Y$

$E_{4.2} = (\lambda u.!())\ A$

$E_5 = \mathsf{bind}\ C = \mathsf{store}() \mathsf{\ in\ } E_6$

$E_6 = e_t\ C$

Given: $(\delta_s, \delta_t) \in \lfloor \Gamma \rfloor_E$

To prove: $(\mathsf{fix} x.e\delta_s, (\mathsf{fix} Y.E_1)()\delta_t) \in \lfloor \mu\ \iota \rfloor_E$

This means from Definition 34 we need to prove that

$\forall^s v.\mathsf{fix} x.e\delta_s \Downarrow {}^s v \implies \exists J', {}^t v_t, {}^t v_f. E_0() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \mu\ \iota \rfloor_V$

This means that given some ${}^s v$ s.t $\mathsf{fix} x.e\delta_s \Downarrow {}^s v$ it suffices to prove that

$$\exists J', {}^t v_t, {}^t v_f. E_0() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \mu\ \iota \rfloor_V \qquad \text{(F-DA-F0)}$$

<u>Claim 1</u>

$\forall 0 \leqslant t < L.\ (e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$

where $\delta'_s = \delta_s \cup \{x \mapsto (\mathsf{fix} x.e)\delta_s\}$ and $\delta'_t = \delta_t \cup \{x \mapsto (\mathsf{fix} x.E_1)\delta_t\}$

We prove this by induction on the recursion tree

<u>Base case</u>: when t is a leaf node

Since for a leaf node $I(t) = 0$ and $x \notin free(e)$ therefore from IH (outer induction) we get

$(e\ \delta_s, e_t\ ()\ \delta_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$

This means from Definition 34 we have

$\forall^s v'.e_s\ \delta_s \Downarrow {}^s v \implies \exists^t v'_t, {}^t v'_f, J'.e_t\ ()\delta_t \Downarrow {}^t v'_t \Downarrow^{J'}\ {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in \lfloor \tau[t/b]\ \iota \rfloor_V$ (BC0)

Since we have to prove $(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$

Therefore from Definition 34 it suffices to prove that

$\forall^s v.e_s\ \delta'_s \Downarrow {}^s v \implies \exists^t v_t, {}^t v_f, J.E_1\ () \Downarrow {}^t v_t \Downarrow^J\ {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau[t/b]\ \iota \rfloor_V$

This means given some ${}^s v$ s.t $e_s\ \delta'_s \Downarrow {}^s v$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J.E_1\ () \Downarrow {}^t v_t \Downarrow^J\ {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau[t/b]\ \iota \rfloor_V$ (BC1)

Instantiating (BC0) with ${}^s v$ we get

$\exists^t v'_t, {}^t v'_f, J'.e_t\ ()\delta'_t \Downarrow {}^t v'_t \Downarrow^{J'}\ {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in \lfloor \tau[t/b]\ \iota \rfloor_V$ (BC2)

From E-release, E-bind, E-subExpE we also know that if

$e_t\ ()\delta_t \Downarrow {}^t v'_t \Downarrow^{J'}\ {}^t v'_f$ then $E_1\ ()\delta'_t \Downarrow {}^t v'_t \Downarrow^{J'}\ {}^t v'_f$

Therefore we get we choose ${}^t v_t, {}^t v_f, J$ as ${}^t v'_t, {}^t v'_f, J'$ in (BC1) and we get the desired from (BC2)

Inductive case: when t is a some internal node

From IH we know that

$\forall 0 \leqslant a < I(t).(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[t'/b]\ \iota \rfloor_E$ where $t' = (t + 1 + \bigtriangleup_b^{t+1,a} I(t))$

Since $\Theta, a, b; \Delta, a < I, b < L; . \vdash \tau[(b + 1 + \bigtriangleup_b^{b+1,a} I)/b] <: \sigma$ therefore from Lemma 38 we know that

$\forall 0 \leqslant a < I(t).(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \sigma\ \iota \rfloor_E$ (F-DA-F0.1)

Claim 2

$(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \sigma\ \iota \rfloor_E \implies ((\mathsf{fix} x.e)\ \delta_s, ((\mathsf{fix} x.(\lambda p.E_2))\ ())\ \delta_t) \in \lfloor \sigma\ \iota \rfloor_E$

Proof is trivial

□

Since from (F-DA-F0.1) we know that

$\forall 0 \leqslant a < I(t).(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \sigma\ \iota \rfloor_E$

Therefore from Claim2 we also get

$\forall 0 \leqslant a < I.(\mathsf{fix} x.e\ \delta_s, \mathsf{fix} x.E_1\ ()\ \delta_t) \in \lfloor \sigma\ \iota \rfloor_E$

Let

$\delta''_s = \delta_s \cup \{x \mapsto \mathsf{fix} x.e\delta_s\}$

$\delta''_t = \delta_t \cup \{x \mapsto ((\mathsf{fix} x.E_1)\delta_t\ ())\}$

From Definition 35 it can been that $(\delta''_s, \delta''_t) \in \lfloor \Gamma, x :_{a<I} \sigma \rfloor_E$

Therefore from IH (outer induction) we get
$(e\ \delta''_s, e_t\ ()\ \delta''_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$

This means from Definition 34 we have
$\forall^s v_0.e_s\ \delta''_s \Downarrow {}^s v_0 \implies \exists J_0, {}^t v_t, {}^t v_f.e_t\ ()\ \ \delta''_t \Downarrow {}^t v_t \Downarrow^{J_0} {}^t v_f \wedge ({}^s v_0, {}^t v_f) \in \lfloor \tau[t/b]\ \iota \rfloor_V$
(F-DA-F1)

In order to prove $(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$ from Definition 34 it suffices to prove
$\forall^s v_s.e\ \delta'_s \Downarrow {}^s v_s \implies \exists J_1, {}^t v'_t, {}^t v_t.E_2[()/p]\delta'_t \Downarrow {}^t v'_t \Downarrow^{J_1} {}^t v_t \wedge ({}^s v_s, {}^t v_t) \in \lfloor \tau[t/b]\ \iota \rfloor_V$

This means given some ${}^s v_s$ s.t $e\ \delta'_s \Downarrow {}^s v_s$ and we need to prove that
$\exists J_1, {}^t v'_t, {}^t v_t.E_2[()/p]\delta'_t \Downarrow {}^t v'_t \Downarrow^{J_1} {}^t v_t \wedge ({}^s v_s, {}^t v_t) \in \lfloor \tau[t/b]\ \iota \rfloor_V$ (F-DA-F2)

From E-release, E-bind, E-subExpE we also know that $E_2[()/p]\delta'_t \xrightarrow{*} e_t[(\mathsf{fix} Y.E_1)\ ()/x]\ ()$ therefore from (F-DA-F1) we get the desired.
This proves Claim1

□

Since from Claim1 we know that $\forall 0 \leqslant t < L.\ (e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[t/b]\ \iota \rfloor_E$. Therefore instantiating it with 0 we get
$(e\ \delta'_s, E_1\ ()\ \delta'_t) \in \lfloor \tau[0/b]\ \iota \rfloor_E$

This means from Definition 34 we have
$\forall^s v'.e\delta'_s \Downarrow {}^s v' \implies \exists^t v'_t, {}^t v'_f, J'.E_1\ ()\delta'_t \Downarrow {}^t v'_t \Downarrow^{J'} {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in \lfloor \tau[0/b]\iota \rfloor_V$

Instantiating it with the given ${}^s v$ and since know that $\mathsf{fix} x.e\delta_s \Downarrow {}^s v$ therefore from E-fix we also know that $e[\mathsf{fix} x.e/x]\delta_s \Downarrow {}^s v$. Hence we have
$\exists^t v'_t, {}^t v'_f, J'.E_1\ ()\delta'_t \Downarrow {}^t v'_t \Downarrow^{J'} {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in \lfloor \tau[0/b]\iota \rfloor_V$ (F-DA-F3)

Since $E_1\ ()\delta'_t \Downarrow {}^t v'_t \Downarrow^{J'} {}^t v'_f$ therefore from E-fix we also know that $\mathsf{fix} x.E_1\ ()\delta_t \Downarrow {}^t v'_t \Downarrow^{J'} {}^t v'_f$.
Also since $\tau[0/b] <: \mu$ therefore from (F-DA-F3) and Lemma 37 we get the desired.

□

Lemma 37. $\forall \Theta, \Delta, \tau, \tau', e_s, e_t, \iota.$

(a) $\Theta; \Delta \vdash \tau <: \tau' \wedge \models \Delta\iota \implies \lfloor \tau\ \iota \rfloor_V \subseteq \lfloor \tau'\ \iota \rfloor_V$

(b) $\Theta; \Delta \vdash [a < I]\tau <: [a < J]\tau' \wedge \models \Delta\iota \implies \lfloor [a < I]\tau\ \iota \rfloor_{NE} \subseteq \lfloor [a < J]\tau'\ \iota \rfloor_{NE}$

Proof. Proof by simultaneous induction on $\Theta; \Delta \vdash \tau <: \tau'$ and $\Theta; \Delta \vdash [a < I]\tau <: [a < J]\tau'$
Proof of statement (a)
We case analyze the different cases:

1. $\multimap$:

$$\frac{\Theta; \Delta \vdash B <: A \qquad \Theta; \Delta \vdash \tau <: \tau'}{\Theta; \Delta \vdash A \multimap \tau <: B \multimap \tau'}$$

To prove: $\lfloor (A \multimap \tau)\ \iota \rfloor_V \subseteq \lfloor (B \multimap \tau')\ \iota \rfloor_V$

This means we need to prove that
$\forall (\lambda x.e, \lambda x.\lambda p.e_t) \in \lfloor A \multimap \tau\ \iota \rfloor_V.(\lambda x.e, \lambda x.\lambda p.e_t) \in \lfloor B \multimap \tau'\ \iota \rfloor_E$

This means given $(\lambda x.e_s, \lambda y.\lambda p.\,\mathsf{let}\,!\,x = y \,\mathsf{in}\, e_t) \in \lfloor A \multimap \tau\ \iota \rfloor_V$ and we need to prove
$(\lambda x.e_s, \lambda y.\lambda p.\,\mathsf{let}\,!\,x = y \,\mathsf{in}\, e_t) \in \lfloor B \multimap \tau'\ \iota \rfloor_V$

This means from Definition 34 we are given that
$\forall e'_s, e'_t.(e'_s, e'_t) \in \lfloor A\ \iota \rfloor_{NE} \implies (e_s[e'_s/x], e_t[e'_t/y][()/p]) \in \lfloor \tau\ \iota \rfloor_E\}$ (SV-A0)

And we need to prove that
$\forall e''_s, e''_t.(e''_s, e'_t) \in \lfloor B\ \iota \rfloor_{NE} \implies (e_s[e''_s/x], e_t[e''_t/y][()/p]) \in \lfloor \tau'\ \iota \rfloor_E\}$

This means given $(e''_s, e'_t) \in \lfloor B\ \iota \rfloor_{NE}$ we need to prove that
$(e_s[e''_s/x], e_t[e''_t/y][()/y]) \in \lfloor \tau'\ \iota \rfloor_E$ (SV-A1)

Since we are given that $(e''_s, e'_t) \in \lfloor B\ \iota \rfloor_{NE}$ therefore from IH (Statement (b)) we have
$(e''_s, e'_t) \in \lfloor A\ \iota \rfloor_{NE}$

In order to prove (SV-A1) we instantiate (SV-A0) with e''_s, e''_t and we get
$(e_s[e''_s/x], e_t[e''_t/y][()/p]) \in \lfloor \tau\ \iota \rfloor_E$
Finally from Lemma 38 we get
$(e_s[e''_s/x], e_t[e''_t/y][()/p]) \in \lfloor \tau'\ \iota \rfloor_E$

Proof of statement (b)

$$\frac{\Theta; \Delta \vdash J \leqslant I \qquad \Theta; \Delta \vdash \tau <: \tau'}{\Theta; \Delta \vdash [a < I]\tau <: [a < J]\tau'}$$

To prove: $\lfloor [a < I]\tau\ \iota \rfloor_{NE} \subseteq \lfloor [a < J]\tau'\ \iota \rfloor_{NE}$

This means we need to prove that
$\forall (e_s, e_t) \in \lfloor [a < I]\tau\ \iota \rfloor_{NE}.(e_s, e_t) \in \lfloor [a < J]\tau'\ \iota \rfloor_{NE}$

This means given $(e_s, e_t) \in \lfloor [a < I]\tau\ \iota \rfloor_{NE}$ and we need to prove
$(e_s, e_t) \in \lfloor [a < J]\tau'\ \iota \rfloor_{NE}$

This means from Definition 34 we are given
$\exists e'_t.e_t = \mathit{coerce}1\ !e'_t\ !() \wedge \forall 0 \leqslant i < I.(e_s, e'_t) \in \lfloor \tau[i/a]\ \iota \rfloor_E$ (SNE0)
and we need to prove
$\exists e''_t.e_t = \mathit{coerce}1\ !e''_t\ !() \wedge \forall 0 \leqslant j < J.(e_s, e''_t) \in \lfloor \tau'[j/a]\ \iota \rfloor_E$ (SNE1)

In order to prove (SNE1) we choose e''_t as e'_t from (SNE0) and we need to prove

$\forall 0 \leqslant j < J.(e_s, e_t') \in \lfloor \tau'[j/a]\ \iota \rfloor_E$

This means given some $0 \leqslant j < J$ and we need to prove that
$(e_s, e_t') \in \lfloor \tau'[j/a]\ \iota \rfloor_E$

From (SNE0) we get
$(e_s, e_t') \in \lfloor \tau[j/a]\ \iota \rfloor_E$

And finally from Lemma 38 we get
$(e_s, e_t'') \in \lfloor \tau'[j/a]\ \iota \rfloor_E$ □

Lemma 38. $\forall \Theta, \Delta, \tau, \tau', e_s, e_t, \iota.$
$\Theta; \Delta \vdash \tau <: \tau' \wedge \models \Delta\iota \implies \lfloor \tau\ \iota \rfloor_E \subseteq \lfloor \tau'\ \iota \rfloor_E$

Proof. Given: $\Theta; \Delta \vdash \tau <: \tau'$
To prove: $\lfloor \tau\ \iota \rfloor_E \subseteq \lfloor \tau'\ \iota \rfloor_E$

It suffices to prove that
$\forall (e_s, e_t) \in \lfloor \tau\ \iota \rfloor_E.(e_s, e_t) \in \lfloor \tau'\ \iota \rfloor_E$

This means given $(e_s, e_t) \in \lfloor \tau\ \iota \rfloor_E$ it suffices to prove that
$(e_s, e_t) \in \lfloor \tau'\ \iota \rfloor_E$

This means from Definition 34 we are given that
$\forall {}^s v_0.e_s \Downarrow {}^s v_0 \implies \exists J_0, {}^t v_0', {}^t v_0.e_t \Downarrow {}^t v_0' \Downarrow^{J_0} {}^t v_0 \wedge ({}^s v_0, {}^t v_0) \in \lfloor \tau\ \iota \rfloor_V$ (S0)

And it suffices to prove that
$\forall {}^s v.e_s \Downarrow {}^s v \implies \exists J, {}^t v_t, {}^t v_f.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau'\ \iota \rfloor_V$

This means given some ${}^s v$ s.t $e_s \Downarrow {}^s v$ and we need to prove
$\exists J, {}^t v_t, {}^t v_f.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge ({}^s v, {}^t v_f) \in \lfloor \tau'\ \iota \rfloor_V$ (S1)

We get the desired from (S0) and Lemma 37

□

1.5.3 Re-deriving dlPCF's soundness

Definition 39 (Closure translation)**.**

$$
\begin{array}{lcl}
(\!|(e, [])|\!) & \triangleq & e \\
(\!|(e, \mathtt{C}_1, \ldots, \mathtt{C}_n)|\!) & \triangleq & \lambda x_1 \ldots x_n.e\ (\!|\mathtt{C}_1|\!)\ \ldots (\!|\mathtt{C}_n|\!)
\end{array}
$$

Definition 40 (Krivine triple translation)**.**

$$
\begin{array}{lcl}
(\!|(e, \rho, \epsilon)|\!) & \triangleq & (\!|(e, \rho)|\!) \\
(\!|(e, \rho, c.\theta)|\!) & \triangleq & (\!|(\!|(e, \rho)|\!)\ (\!|c|\!), ., \theta|\!)
\end{array}
$$

Lemma 41 (Type preservation for Closure translation)**.** $\forall \Theta, \Delta, e, \rho, \tau.$
$\Theta; \Delta \vdash_J (e, \rho) : \sigma \implies \Theta; \Delta; . \vdash_J (\!|(e, \rho)|\!) : \sigma$

Proof.

$$
\frac{\begin{array}{c}\Theta; \Delta; x_1 : [a < I_1]\tau_1 \ldots x_n : [a < I_n]\tau_n \vdash_K e : \sigma \\ \Theta, a; \Delta, a < I_i \vdash_{H_i} \mathtt{C}_i : \tau_i \qquad J \geqslant K + I_1 + \ldots + I_n + \sum_{a<I_1} H_1 + \ldots + \sum_{a<I_n} H_n\end{array}}{\Theta; \Delta \vdash_J (e, (\mathtt{C}_1 \ldots \mathtt{C}_n)) : \sigma}
$$

$J' = K + I_1 + \ldots + I_n + \sum_{a<I_1} H_1 + \ldots + \sum_{a<I_n} H_n$
D1:

$$\frac{}{\Theta, a; \Delta; .a < I_i \vdash_{H_i} (\!|\mathtt{C}_i|\!) : \tau_i}\ \text{IH}$$

D0:

$$\frac{\dfrac{}{\Theta; \Delta; x_1 : [a_1 < I_1]\tau_1, \ldots, x_n : [a_n < I_n]\tau_n \multimap\!\vdash_K e : \sigma}\ \text{Given}}{\Theta; \Delta; . \vdash_K \lambda x_1 \ldots x_n.e : [a_1 < I_1]\tau_1 \multimap [a_2 < I_2]\tau_1 \multimap \ldots [a_n < I_n]\tau_n \multimap \sigma}\ \text{D-lam}$$

Main derivation:

$$\frac{\dfrac{\dfrac{D0 \qquad D1}{\Theta; \Delta; . \vdash_{J'} \lambda x_1 \ldots x_n.e\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!) : \sigma}\ \text{D-app}}{\Theta; \Delta; . \vdash_J \lambda x_1 \ldots x_n.e\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!) : \sigma}\ \text{Lemma 3.5 of [3]}}{\Theta; \Delta; . \vdash_J (\!|(e, \mathtt{C}_1 \ldots \mathtt{C}_n)|\!) : \sigma}\ \text{Definition 39}$$

□

Theorem 42 (Type preservation for Krivine triple translation). $\forall \Theta, \Delta, e, \rho, \theta, \tau.$
$\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau \implies \Theta; \Delta; . \vdash_I (\!|(e, \rho, \theta)|\!) : \tau$

Proof.

$$\frac{\Theta; \Delta \vdash_K (e, \rho) : \sigma \qquad \Theta; \Delta \vdash_J \theta : (\sigma, \tau) \qquad I \geqslant K + J}{\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau}$$

Let $I' = K + J$
Proof by induction on θ

1. Case ϵ:

   Given: $\Theta; \Delta \vdash_I (e, \rho, \epsilon) : \tau$

   To prove: $\Theta; \Delta; . \vdash_I (\!|(e, \rho, \epsilon)|\!) : \tau$

   D0:

   $$\frac{}{\Theta; \Delta; . \vdash_K (\!|(e, \rho)|\!) : \sigma}\ \text{Lemma 41}$$

   Main derivation:

   $$\frac{\dfrac{\dfrac{D0}{\Theta; \Delta; . \vdash_{I'} (\!|(e, \rho)|\!) : \tau}\ \text{Lemma 3.5 of [3]}}{\Theta; \Delta; . \vdash_{I'} (\!|(e, \rho, \epsilon)|\!) : \tau}\ \text{Definition 40}}{\Theta; \Delta; . \vdash_I (\!|(e, \rho, \epsilon)|\!) : \tau}\ \text{Lemma 3.5 of [3]}$$

2. Case $\mathtt{C}.\theta'$:

   Given: $\Theta; \Delta \vdash_I (e, \rho, \mathtt{C}.\theta') : \tau$

   To prove: $\Theta; \Delta; . \vdash_I (\!|(e, \rho, \mathtt{C}.\theta')|\!) : \tau$

   Since $\theta = \mathtt{C}.\theta'$ therefore from dlPCF's type rule for $\mathtt{C}.\theta'$ we know that $\sigma = [d < L]\gamma \multimap \mu$

That is we are given that

$$\frac{\Theta, d; \Delta, d < L_g \vdash_{K_g} \mathtt{C} : \gamma \qquad \Theta; \Delta \vdash_{H_g} \theta' : (\mu, \tau) \qquad J \geqslant H_g + \sum_{d<L_g} K_g + L_g}{\Theta; \Delta \vdash_J \mathtt{C}.\theta' : ([d < L_g]\gamma \multimap \mu, \tau)}$$

D2:

$$\dfrac{\dfrac{}{\Theta; \Delta \vdash_J \mathtt{C}.\theta' : ([d < L_g]\gamma \multimap \mu, \tau)}\ \text{Given}}{\Theta; \Delta \vdash_{H_g} \theta' : (\mu, \tau)}\ \text{By inversion}$$

D1:

$$\dfrac{}{\Theta; \Delta; . \vdash_K (\!|(e, \rho)|\!) : [d < L_g]\gamma \multimap \mu}\ \text{Lemma 41}$$

D0:

$$\dfrac{D1 \qquad \dfrac{\dfrac{}{\Theta, d; \Delta, d < L_g \vdash_{K_g} \mathtt{C} : \gamma}\ \text{Given}}{\Theta, d; \Delta, d < L_g \vdash_{K_g} (\!|\mathtt{C}|\!) : \gamma}\ \text{Lemma 41}}{\Theta; \Delta; . \vdash_{K+L_g+\sum_{L_g} K_g} (\!|(e, \rho)|\!)\ (\!|\mathtt{C}|\!) : \mu}\ \text{D-app}$$

D0.1:

$$\dfrac{D0}{\Theta; \Delta; . \vdash_{K+L_g+\sum_{L_g} K_g} ((\!|(e, \rho)|\!)\ (\!|\mathtt{C}|\!), .) : \mu}$$

D0.0:

$$\dfrac{\dfrac{D0.1 \qquad D2}{\Theta; \Delta \vdash_{K+L_g+\sum_{L_g} K_g+H_g} ((\!|(e, \rho)|\!)\ (\!|\mathtt{C}|\!), ., \theta') : \tau} \qquad J \geqslant L_g + \sum_{L_g} K_g + H_g}{\Theta; \Delta \vdash_{K+J} ((\!|(e, \rho)|\!)\ (\!|\mathtt{C}|\!), ., \theta') : \tau}\ \text{Lemma 3.5 of [3]}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{D0.0}{\Theta; \Delta; . \vdash_{I'} (\!|(e, \rho)\ (\!|\mathtt{C}|\!), ., \theta|\!) : \tau}\ \text{IH}}{\Theta; \Delta; . \vdash_{I'} (\!|(e, \rho, \mathtt{C}.\theta)|\!) : \tau}\ \text{Definition 40}}{\Theta; \Delta; . \vdash_I (\!|(e, \rho, \mathtt{C}.\theta)|\!) : \tau}\ \text{Lemma 3.5 of [3]}$$

$\square$

Definition 43 (Equivalence for λ-amor)**.**

$$v_1 \overset{s}{\approx}_{aV} v_2 \triangleq \begin{cases} True & v_1 = () \wedge v_2 = () \\ \forall e', e'', s' < s.e' \overset{s'}{\approx}_{aE} e'' \implies & v_1 = \lambda x.e_2 \wedge v_2 = \lambda x.e_2 \\ e_1[e'/x] \overset{s'}{\approx}_{aE} e_2[e''/x] & \\ e_1 \overset{s}{\approx}_{aE} e_2 & v_1 = !e_1 \wedge v_2 = !e_2 \\ & v_1 = \Lambda.e_1 \wedge v_2 = \Lambda.e_2 \\ \forall i < s.v_1 \Downarrow_i^k v_a \implies & v_1 = \mathsf{ret} - \wedge v_2 = \mathsf{ret} - \\ v_2 \Downarrow^k v_b \wedge v_a \overset{s-i}{\approx}_{aE} v_b & v_1 = \mathsf{bind} - = - \mathsf{in} - \wedge v_2 = \mathsf{bind} - = - \mathsf{in} - \\ & v_1 = \uparrow^n \wedge v_2 = \uparrow^n \\ & v_1 = \mathsf{release} - = - \mathsf{in} - \wedge v_2 = \mathsf{release} - = - \mathsf{in} - \\ & v_1 = \mathsf{store} - \wedge v_2 = \mathsf{store} - \\ v_{a1} \overset{s}{\approx}_{aV} v_{b1} \wedge v_{a2} \overset{s}{\approx}_{aV} v_{b2} & v_1 = \langle\!\langle v_{a1}, v_{a2} \rangle\!\rangle \wedge v_2 = \langle\!\langle v_{b1}, v_{b2} \rangle\!\rangle \\ v_{a1} \overset{s}{\approx}_{aV} v_{b1} \wedge v_{a2} \overset{s}{\approx}_{aV} v_{b2} & v_1 = \langle v_{a1}, v_{a2} \rangle \wedge v_2 = \langle v_{b1}, v_{b2} \rangle \\ v_a \overset{s}{\approx}_{aV} v_b & v_1 = \mathsf{inl}(v_a) \wedge v_2 = \mathsf{inl}(v_b) \\ v_a \overset{s}{\approx}_{aV} v_b & v_1 = \mathsf{inr}(v_a) \wedge v_2 = \mathsf{inr}(v_b) \end{cases}$$

$$e_1 \overset{s}{\approx}_{aE} e_2 \triangleq \forall i < s.e_1 \Downarrow_i v_a \implies e_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

$$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \triangleq dom(\delta_1) = dom(\delta_2) \wedge \forall x \in dom(\delta_1).\delta_1(x) \overset{s}{\approx}_{aE} \delta_2(x)$$

Lemma 44 (Monotonicity lemma for value equivalence)**.** $\forall v_1, v_2, s.$
$v_1 \overset{s}{\approx}_{aV} v_2 \implies \forall s' < s.v_1 \overset{s'}{\approx}_{aV} v_2$

Proof. Given: $v_1 \overset{s}{\approx}_{aV} v_2$

To prove: $\forall s' < s.v_1 \overset{s'}{\approx}_{aV} v_2$

This means given some $s' < s$ and it suffices to prove that $v_1 \overset{s'}{\approx}_{aV} v_2$

We induct on v_1

1. $v_1 = ()$:

   Since we are given that $v_1 \overset{s}{\approx}_{aV} v_2$ therefore we get the desired Directly from Definition 43

2. $v_1 = \lambda x.e_1$:

   Since we are given that $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we are given that

   $$\forall e', e'', s'' < s.e' \overset{s''}{\approx}_{aE} e'' \implies e_1[e'/x] \overset{s''}{\approx}_{aE} e_2[e''/x] \qquad \text{(M-L0)}$$

   and we need to prove that $v_1 \overset{s'}{\approx}_{aV} v_2$ therefore again from Definition 43 we need to prove that

   $$\forall e_1', e_1'', s_1'' < s.e_1' \overset{s_1''}{\approx}_{aE} e_1'' \implies e_1[e_1'/x] \overset{s_1''}{\approx}_{aE} e_2[e_1''/x]$$

   This means given some $e_1', e_1'', s_1'' < s'$ s.t $e_1' \overset{s_1''}{\approx}_{aE} e_1''$ we need to prove that

   $$e_1[e_1'/x] \overset{s_1''}{\approx}_{aE} e_2[e_1''/x]$$

   Instantiating (M-L0) with e_1', e_1'', s_1'' we get $e_1[e_1'/x] \overset{s_1''}{\approx}_{aE} e_2[e_1''/x]$

3. $v_1 = !e_1$:

   Since we are given $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we have $e_1 \overset{s}{\approx}_{aE} e_2$ where $v_2 = !e_2$

   Similarly from Definition 43 it suffices to prove that $e_1 \overset{s'}{\approx}_{aE} e_2$

   We get this directly from Lemma 45

4. $v_1 = \Lambda e_1$:

   Similar reasoning as in the $!e_1$ case

5. $v_1 = \mathsf{ret}\, e_1$:

   Since we are given $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we have

   $$\forall i < s.v_1 \Downarrow_i^k v_a \implies v_2 \Downarrow^k v_b \wedge v_a \overset{s-i}{\approx}_{aE} v_b \text{ where } v_2 = \mathsf{ret}\, e_2 \tag{MV-R0}$$

   Similarly from Definition 43 it suffices to prove that

   $\forall j < s'.v_1 \Downarrow_i^k v_a \implies v_2 \Downarrow^k v_b \wedge v_a \overset{s'-j}{\approx}_{aE} v_b$

   This means given some $j < s'$ and $v_1 \Downarrow_i^k v_a$ and it suffices to prove that

   $v_2 \Downarrow^k v_b \wedge v_a \overset{s'-j}{\approx}_{aE} v_b$

   Instantiating (MV-R0) with j we get $v_2 \Downarrow^k v_b \wedge v_a \overset{s-j}{\approx}_{aE} v_b$

   Since we have $v_a \overset{s-j}{\approx}_{aE} v_b$ therefore from Lemma 45 we also get $v_a \overset{s'-j}{\approx}_{aE} v_b$

6. $v_1 = \mathsf{bind} - = - \mathsf{in} -, \uparrow^n, \mathsf{release} - = - \mathsf{in} -, \mathsf{store} -$:

   Similar reasoning as in the $\mathsf{ret} -$ case

7. $v_1 = \langle\langle v_{a1}, v_{a2} \rangle\rangle$:

   From Definition 43 and IH we get the desired

8. $v_1 = \langle v_{a1}, v_{a2} \rangle$:

   From Definition 43 and IH we get the desired

9. $v_1 = \mathsf{inl}(v)$:

   From Definition 43 and IH we get the desired

10. $v_1 = \mathsf{inr}(v)$:

    From Definition 43 and IH we get the desired

$\square$

Lemma 45 (Monotonicity lemma for expression equivalence)**.** $\forall e_1, e_2, s.$
$e_1 \overset{s}{\approx}_{aE} e_2 \implies \forall s' < s.e_1 \overset{s'}{\approx}_{aE} e_2$

Proof. Given: $e_1 \overset{s}{\approx}_{aE} e_2$

To prove: $\forall s' < s.e_1 \overset{s'}{\approx}_{aE} e_2$

This means given some $s' < s$ and we need to prove $e_1 \overset{s'}{\approx}_{aE} e_2$

Since we are given $e_1 \overset{s}{\approx}_{aE} e_2$ therefore from Definition 43 we have

$\forall i < s.e_1 \Downarrow_i v_a \implies e_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (ME0)

Similarly from Definition 43 it suffices to prove that

$\forall j < s'.e_1 \Downarrow_j v_a \implies e_2 \Downarrow v_b \wedge v_a \overset{s'-j}{\approx}_{aV} v_b$

This means given some $j < s'$ s.t $e_1 \Downarrow_j v_a$ and we need to prove

$e_2 \Downarrow v_b \wedge v_a \overset{s'-j}{\approx}_{aV} v_b$

We get the desired from (ME0) and Lemma 44

□

Lemma 46 (Monotonicity lemma for δ equivalence)**.** $\forall \delta_1, \delta_2, s.$

$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \implies \forall s' < s.\delta_1 \overset{s'}{\approx}_{aE} \delta_2$

Proof. From Definition 43 and Lemma 45 □

Theorem 47 (Fundamental theorem for equivalence relation of λ-amor)**.** $\forall \delta_1, \delta_2, e, s.$

$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \implies e\delta_1 \overset{s}{\approx}_{aE} e\delta_2$

Proof. We induct on e

1. $e = x$:

   We need to prove that $x\delta_1 \overset{s}{\approx}_{aE} x\delta_2$

   This means it suffices to prove that $\delta_1(x) \overset{s}{\approx}_{aE} \delta_2(x)$

   We get this directly from Definition 43

2. $e = \lambda y.e'$:

   We need to prove that $\lambda y.e'\delta_1 \overset{s}{\approx}_{aE} \lambda y.e'\delta_2$

   This means from Definition 43 it suffices to prove that

   $\forall i < s.\lambda y.e'\delta_1 \Downarrow_i v_a \implies \lambda y.e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

   This means that given some $i < s$ s.t $\lambda y.e'\delta_1 \Downarrow_i v_a$ it suffices to prove that

   $\lambda y.e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-L0)

   From E-val we know that $v_a = \lambda y.e'\delta_1$

   From (FTE-L0) we need to prove that

   (a) $\lambda y.e'\delta_2 \Downarrow v_b$:

      From E-val we know that $v_b = \lambda y.e'\delta_2$

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:
We need to prove that
$\lambda y.e'\delta_1 \overset{s}{\approx}_{aV} \lambda y.e'\delta_2$

This means from Definition 43 it suffices to prove that
$\forall e'_1, e'_2, s' < s. e'_1 \overset{s'}{\approx}_{aE} e'_2 \implies e'\delta_1[e'_1/y] \overset{s'}{\approx}_{aE} e'\delta_2[e'_2/y]$

This further means that given some $e'_1, e'_2, s' < s$ s.t $e'_1 \overset{s'}{\approx}_{aE} e'_2$ it suffices to prove that
$e'\delta_1[e'_1/y] \overset{s'}{\approx}_{aE} e'\delta_2[e'_2/y]$

We get this from IH and Lemma 46

3. $e = \mathsf{fix} y.e'$:
We induct on s
IHi: $\forall s'' < s.\ \delta_1 \overset{s''}{\approx}_{aE} \delta_2 \implies \mathsf{fix} y.e'\delta_1 \overset{s'}{\approx}_{aE} \mathsf{fix} y.e'\delta_2$

To prove: $\delta_1 \overset{s}{\approx}_{aE} \delta_2 \implies \mathsf{fix} y.e'\delta_1 \overset{s}{\approx}_{aE} \mathsf{fix} y.e'\delta_2$
This means we are given $\delta_1 \overset{s}{\approx}_{aE} \delta_2$ and we need to prove
$\mathsf{fix} y.e'\delta_1 \overset{s}{\approx}_{aE} \mathsf{fix} y.e'\delta_2$

From Definition 43 it suffices to prove that
$\forall i < s.\mathsf{fix} y.e'\delta_1 \Downarrow_i v_a \implies \mathsf{fix} y.e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This means given some $i < s$ s.t $\mathsf{fix} y.e'\delta_1 \Downarrow_i v_a$ and we need to prove $\mathsf{fix} y.e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

Since we are given that $\mathsf{fix} y.e'\delta_1 \Downarrow_i v_a$ therefore from E-fix we know that
$e'[\mathsf{fix} x.e'\delta_1/y]\delta_1 \Downarrow_{i-1} v_a$

Instantiating IHi with $s-1$ and using Lemma 46 we get
$\mathsf{fix} y.e'\delta_1 \overset{s-1}{\approx}_{aE} \mathsf{fix} y.e'\delta_2 \qquad \text{(F1)}$

Let
$\delta'_1 = \delta_1 \cup \{y \mapsto \mathsf{fix} y.e'\delta_1\}$
$\delta'_2 = \delta_2 \cup \{y \mapsto \mathsf{fix} y.e'\delta_2\}$

From Lemma 46 and (F1) we know that $\delta'_1 \overset{s-1}{\approx}_{aE} \delta'_2$

Therefore from IH of outer induction we know that we have
$e'\delta'_1 \overset{s-1}{\approx}_{aE} e'\delta'_2$

This means from Definition 43 we know that
$\forall i' < (s-1).e'\delta'_1 \Downarrow_{i'} v_a \implies e'\delta'_2 \Downarrow v_b \wedge v_a \overset{s-1-i'}{\approx}_{aV} v_b$

Instantiating with $i-1$ and since we know that $e'\delta'_1 \Downarrow_{i-1} v_a$ and therefore we get
$e'\delta'_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ which is the desired.

4. $e = e_1\ e_2$:

   We need to prove that $e_1\ e_2\delta_1 \overset{s}{\approx}_{aE} e_1\ e_2\delta_2$

   This means from Definition 43 it suffices to prove that

   $\forall i < s.e_1\ e_2\delta_1 \Downarrow_i v_a \implies e_1\ e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

   This means that given some $i < s$ s.t $e_1\ e_2\delta_1 \Downarrow_i v_a$ it suffices to prove that

   $e_1\ e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-A0)

   <u>IH1</u>: $e_1\delta_1 \overset{s}{\approx}_{aE} e_1\delta_2$

   Therefore from Definition 43 we have

   $\forall j < s.e_1\delta_1 \Downarrow_j v'_a \implies e_1\delta_2 \Downarrow v'_b \wedge v'_a \overset{s-j}{\approx}_{aV} v'_b$ (FTE-A1)

   Since $(e_1\delta_1\ e_2\delta_1) \Downarrow_i v_a$ therefore from E-app we know that $\exists i_1 < i.e_1\delta_1 \Downarrow_{i_1} \lambda y.e'$

   Therefore instantiating (FTE-A1) with i_1 we get $e_1\delta_2 \Downarrow v'_b \wedge v'_a \overset{s-i_1}{\approx}_{aV} v'_b$ (FTE-A1.1)

   Since $v'_a = \lambda y.e'$ and since $v'_a \overset{s-i_1}{\approx}_{aV} v'_b$ therefore from Definition 43 we know that $v'_b = \lambda y.e''$

   Again since $\lambda y.e' \overset{s-i_1}{\approx}_{aV} \lambda y.e''$ therefore from Definition 43 we know that

   $\forall e'_1, e'_2, s' < (s - i_1).e'_1 \overset{s'}{\approx}_{aE} e'_2 \implies e'[e'_1/y] \overset{s'}{\approx}_{aE} e''[e'_2/y]$ (FTE-A2)

   <u>IH2</u>: $e_2\delta_1 \overset{s-i_1-1}{\approx}_{aE} e_2\delta_2$

   Instantiating (FTE-A2) with $e_2\delta_1, e_2\delta_2$ we get

   $e'[e_2\delta_1/y] \overset{s-i_1-1}{\approx}_{aE} e''[e_2\delta_1/y]$

   Again from Definition 43 we have

   $\forall j < (s - i_1 - 1).e'[e_2\delta_1/y] \Downarrow_j v''_a \implies e''[e_2\delta_1/y] \Downarrow v''_b \wedge v''_a \overset{s-i_1-1-j}{\approx}_{aV} v''_b$ (FTE-A2.1)

   Since $(e_1\delta_1\ e_2\delta_1) \Downarrow_i v_a$ therefore from E-app we know that $\exists i_2 = i - i_1 - 1.e'[e_2\delta_1/x] \Downarrow_{i_2} v_a$

   Instantiating (FTE-A2.1) with i_2 we get $e''[e_2\delta_1/y] \Downarrow v''_b \wedge v_a \overset{s-i_1-1-i_2}{\approx}_{aV} v''_b$

   Since $i = i_1 + i_2 + 1$ therefore this proves (FTE-A0) and we are done.

5. $e = \langle\langle e_1, e_2 \rangle\rangle$:

   We need to prove that $\langle\langle e_1, e_2\rangle\rangle\delta_1 \overset{s}{\approx}_{aE} \langle\langle e_1, e_2\rangle\rangle\delta_2$

   This means from Definition 43 it suffices to prove that

   $\forall i < s.\langle\langle e_1, e_2\rangle\rangle\delta_1 \Downarrow_i v_a \implies \langle\langle e_1, e_2\rangle\rangle\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

   This means that given some $i < s$ s.t $\langle\langle e_1, e_2\rangle\rangle\delta_1 \Downarrow_i v_a$ it suffices to prove that

   $\langle\langle e_1, e_2\rangle\rangle\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-TI0)

From E-TI we know that $v_a = \langle\!\langle v_{a1}, v_{a2}\rangle\!\rangle$ and $e_1\delta_1 \Downarrow_{i_1} v_{a1}$ and $e_2\delta_1 \Downarrow_{i_2} v_{a2}$

<u>IH1</u>: $e_1\delta_1 \overset{s}{\approx}_{aE} e_1\delta_2$

Therefore from Definition 43 we have

$\forall i < s.e_1\delta_1 \Downarrow_i v_{a1} \implies e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$

Since we know that $e_1\delta_1 \Downarrow_{i_1} v_{a1}$ therefore we get

$e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$ (FTE-TI1)

<u>IH2</u>: $e_2\delta_1 \overset{s}{\approx}_{aE} e_2\delta_2$

Similarly from Definition 43 we have

$\forall i < s.e_2\delta_1 \Downarrow_i v_{a1} \implies e_2\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$

Since we know that $e_2\delta_1 \Downarrow_{i_2} v_{a2}$ therefore we get

$e_2\delta_2 \Downarrow v_{b2} \wedge v_{a2} \overset{s-i_2}{\approx}_{aV} v_{b2}$ (FTE-TI2)

From (FTE-TI0) we need to prove

(a) $\langle\!\langle e_1, e_2\rangle\!\rangle\delta_2 \Downarrow v_b$:
We get this from (FTE-TI1), (FTE-TI2) and E-TI

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:
Since $i = i_1 + i_2$, $v_a = \langle\!\langle v_{a1}, v_{a2}\rangle\!\rangle$ and $v_b = \langle\!\langle v_{b1}, v_{b2}\rangle\!\rangle$ it suffices to prove that
$\langle\!\langle v_{a1}, v_{a2}\rangle\!\rangle \overset{s-i_1-i_2}{\approx}_{aV} \langle\!\langle v_{b1}, v_{b2}\rangle\!\rangle$

From Definition 43 it suffices to prove that
$v_{a1} \overset{s-i_1-i_2}{\approx}_{aV} v_{b1}$ and $v_{a2} \overset{s-i_1-i_2}{\approx}_{aV} v_{b2}$
We get this from (FTE-TI1), (FTE-TI2) and Lemma 44

6. $e = \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2$:

We need to prove that $\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_1 \overset{s}{\approx}_{aE} \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_2$

This means from Definition 43 it suffices to prove that

$\forall i < s.\, \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_1 \Downarrow_i v_a \implies \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This means that given some $i < s$ s.t $\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_1 \Downarrow_i v_a$ it suffices to prove that

$\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-TE0)

<u>IH1</u>: $e_1\delta_1 \overset{s}{\approx}_{aE} e_1\delta_2$

Therefore from Definition 43 we have

$\forall i < s.e_1\delta_1 \Downarrow_i v_{a1} \implies e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$

Since we know that $\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e_1 \mathsf{\ in\ } e_2\delta_1 \Downarrow_i v_a$ therefore from E-TE we know that $\exists i_1 < s.e_1\delta_1 \Downarrow_{i_1} \langle\!\langle v'_{a1}, v'_{a2}\rangle\!\rangle$. Therefore we get

$e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$ (FTE-TE1)

Since $v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$ and $v_{a1} = \langle\langle v'_{a1}, v'_{a2}\rangle\rangle$ therefore from Definition 43 we have
$v_{b1} = \langle\langle v'_{b1}, v'_{b2}\rangle\rangle$ (FTE-TE1.1)

Let
$\delta'_1 = \delta_1 \cup \{x \mapsto \langle\langle v'_{a1}, v'_{a2}\rangle\rangle\}$
$\delta'_2 = \delta_2 \cup \{x \mapsto \langle\langle v'_{b1}, v'_{b2}\rangle\rangle\}$
<u>IH2</u>: $e_2\delta'_1 \overset{s-i_1}{\approx}_{aE} e_2\delta'_2$
Therefore from Definition 43 we have
$\forall i < (s - i_1).e_2\delta'_1 \Downarrow_i v_a \implies e_2\delta'_2 \Downarrow v_{b2} \wedge v_a \overset{s-i_1-i}{\approx}_{aV} v_b$

Since we know that $\mathsf{let}\langle\langle x, y\rangle\rangle = e_1$ in $e_2\delta_1 \Downarrow_i v_a$ therefore from E-TE we know that $\exists i_2 = i - i_1.e_2\delta'_1 \Downarrow_{i_2} v_a$. Therefore we get
$e_2\delta'_2 \Downarrow v_{b2} \wedge v_a \overset{s-i_1-i_2}{\approx}_{aV} v_b$ (FTE-TE2)

This proves the desired

7. $e = \langle e_{a1}, e_{a2}\rangle$:

   Similar reasoning as in the $\langle\langle e_{a1}, e_{a2}\rangle\rangle$ case above

8. $e = \mathsf{fst}(e')$:

   We need to prove that $\mathsf{fst}(e')\delta_1 \overset{s}{\approx}_{aE} \mathsf{fst}(e')\delta_2$

   This means from Definition 43 it suffices to prove that

   $\forall i < s.\mathsf{fst}(e')\delta_1 \Downarrow_i v_a \implies \mathsf{fst}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

   This means that given some $i < s$ s.t $\mathsf{fst}(e')\delta_1 \Downarrow_i v_a$ it suffices to prove that

   $\mathsf{fst}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-F0)

   Since we know that $\mathsf{fst}(e')\delta_1 \Downarrow_i v_a$ therefore from E-fst we know that $e'\delta_1 \Downarrow_i \langle\langle v_a, -\rangle\rangle$
   <u>IH</u>: $e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$
   This means from Definition 43 we have
   $\forall j < s.e'\delta_1 \Downarrow_j v_{a1} \implies e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$

   Instantiating with i we get $e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$

   Since we know that $v_{a1} = \langle\langle v_a, -\rangle\rangle$ therefore from Definition 43 we also know that
   $v_{b1} = \langle\langle v_b, -\rangle\rangle$ s.t $v_a \overset{s}{\approx}_{aV} v_b$
   This proves the desired.

9. $e = \mathsf{snd}(e')$:

   Similar reasoning as in the $\mathsf{fst}(e')$ case

10. $e = \mathsf{inl}(e')$:

    We need to prove that $\mathsf{inl}(e')\delta_1 \overset{s}{\approx}_{aE} \mathsf{inl}(e')\delta_2$

    This means from Definition 43 it suffices to prove that

    $\forall i < s. \mathsf{inl}(e')\delta_1 \Downarrow_i v_a \implies \mathsf{inl}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

    This means that given some $i < s$ s.t $\mathsf{inl}(e')\delta_1 \Downarrow_i v_a$ it suffices to prove that

    $\mathsf{inl}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-IL0)

    Since we know that $\mathsf{inl}(e')\delta_1 \Downarrow_i v_a$ therefore from E-inl we know that $v_a = \mathsf{inl}(()v'_a)$ and $e'\delta_1 \Downarrow_i v'_a$

    <u>IH</u>: $e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$

    This means from Definition 43 we have

    $\forall j < s. e'\delta_1 \Downarrow_j v_{a1} \implies e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$

    Instantiating with i we get $e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$

    Since $e'\delta_2 \Downarrow v_{b1}$ therefore from E-inl we have $\mathsf{inl}(e')\delta_2 \Downarrow \mathsf{inl}(v_{b1})$

    And since we know that $v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$ therefore from Definition 43 we also know that $\mathsf{inl}(v_{a1}) \overset{s-i}{\approx}_{aV} \mathsf{inl}(v_{b1})$

    This proves the desired.

11. $e = \mathsf{inr}(e')$:

    Similar reasoning as in the $\mathsf{inl}(e')$ case

12. $e = \mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r$:

    We need to prove that $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \overset{s}{\approx}_{aE} \mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_2$

    This means from Definition 43 it suffices to prove that

    $\forall i < s. \mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \Downarrow_i v_a \implies \mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

    This means that given some $i < s$ s.t $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \Downarrow_i v_a$ it suffices to prove that

    $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-C0)

    Since we know that $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \Downarrow_i v_a$ therefore two cases arise:

    2 cases arise:

    (a) $e_c\delta_1 \Downarrow \mathsf{inl}(v_{c1})$:

        <u>IH1</u> $e_c\delta_1 \overset{s}{\approx}_{aE} e_c\delta_2$

        This means from Definition 43 we have

        $\forall j < s. e_c\delta_1 \Downarrow_j v_{c1} \implies e_c\delta_2 \Downarrow v_{c2} \wedge v_{c1} \overset{s-j}{\approx}_{aV} v_{c2}$

        Since we know that $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \Downarrow_i v_a$ therefore from E-case1 we know that $\exists i_1$ s.t $e_c\delta_1 \Downarrow_{i_1} \mathsf{inl}(v'_{c1})$

Therefore instantiating with i_1 we get $e_c\delta_2 \Downarrow v_{c2} \wedge v_{c1} \overset{s-i_1}{\approx}_{aV} v_{c2}$

From Definition 43 we know that $\exists v'_{c2}.v_{c2} = \mathsf{inl}(v'_{c2})$ s.t $v'_{c1} \overset{s-i_1}{\approx}_{aV} v'_{c2}$

<u>IH2</u> $e_l\delta_1[v'_{c1}/x] \overset{s-i_1}{\approx}_{aE} e_l\delta_2[v'_{c2}/x]$

This means from Definition 43 we have
$\forall j < (s - i_1).e_l\delta_1[v'_{c1}/x] \Downarrow_j v_{l1} \implies e_l\delta_2[v'_{c2}/x] \Downarrow v_{l2} \wedge v_{l1} \overset{s-i_1-j}{\approx}_{aV} v_b$

Since we know that $\mathsf{case}\ e_c\ \mathsf{of}\ e_l; e_r\delta_1 \Downarrow_i v_a$ therefore from E-case1 we know that $\exists i_2$ s.t $e_l\delta_1 \Downarrow_{i_2} v_a$

Therefore instantiating with i_2 we get $e_l\delta_2[v'_{c2}/x] \Downarrow v_{l2} \wedge v_a \overset{s-i_1-j}{\approx}_{aV} v_b$

This proves the desired

(b) $e_c\delta_1 \Downarrow \mathsf{inr}(v_{c1})$:
Similar reasoning as in the previous case

13. $e = !e'$:

We need to prove that $!e'\delta_1 \overset{s}{\approx}_{aE} !e'\delta_2$

This means from Definition 43 it suffices to prove that

$\forall i < s.!e'\delta_1 \Downarrow_i v_a \implies !e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This means that given some $i < s$ s.t $!e'\delta_1 \Downarrow_i v_a$ it suffices to prove that

$!e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-B0)

From E-val we know that $v_a = !e'\delta_1$ and $i = 0$

<u>IH</u>: $e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$

From (FTE-B0) we need to prove that

(a) $!e'\delta_2 \Downarrow v_b$:
From E-val we know that $v_b = !e'\delta_2$

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:
We need to prove that
$!e'\delta_1 \overset{s}{\approx}_{aV} !e'\delta_2$

This means from Definition 43 it suffices to prove that
$e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$

We get this directly from IH

14. $e = \mathsf{let}\,!\,x = e'_1\ \mathsf{in}\ e'_2$:

We need to prove that $\mathsf{let}\,!\,x = e'_1\ \mathsf{in}\ e'_2\delta_1 \overset{s}{\approx}_{aE} \mathsf{let}\,!\,x = e'_1\ \mathsf{in}\ e'_2\delta_2$

This means from Definition 43 it suffices to prove that

$\forall i < s.\,\mathsf{let}\,!\,x = e'_1\ \mathsf{in}\ e'_2\delta_1 \Downarrow_i v_a \implies \mathsf{let}\,!\,x = e'_1\ \mathsf{in}\ e'_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This means that given some $i < s$ s.t $\mathsf{let}\,!\,x = e_1'$ in $e_2'\delta_1 \Downarrow_i v_a$ it suffices to prove that

$\mathsf{let}\,!\,x = e_1'$ in $e_2'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-BE0)

<u>IH1</u>: $e_1'\delta_1 \overset{s}{\approx}_{aE} e_1'\delta_2$

This means from Definition 43 we have

$\forall j < s.e_1'\delta_1 \Downarrow_j v_{a11} \implies e_1'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b11}$

Since we know that $\mathsf{let}\,!\,x = e_1'$ in $e_2'\delta_1 \Downarrow_i v_a$ therefore from E-subExpE we know that $\exists i_1.e_1'\delta_1 \Downarrow_{i_1} !e_{b1}$

Instantiating with i_1 we get $e_1'\delta_2 \Downarrow v_{b11} \wedge v_{a11} \overset{s-i_1}{\approx}_{aV} v_{b11}$

Since we know that $v_{a11} =!e_{b1}$ therefore from Definition 43 we also know that

$v_{b11} =!e_{b2}$ s.t $e_{b1} \overset{s-i_1}{\approx}_{aE} e_{b2}$

<u>IH2</u>: $e_2'[e_{b1}/x]\delta_1 \overset{s-i_1}{\approx}_{aE} e_2'[e_{b2}/x]\delta_2$

This means from Definition 43 we have

$\forall j < s.e_2'[e_{b1}/x]\delta_1 \Downarrow_j v_a \implies e_2'[e_{b2}/x]\delta_2 \Downarrow v_b \wedge v_a \overset{s-i_1-j}{\approx}_{aV} v_b$

Since we know that $\mathsf{let}\,!\,x = e_1'$ in $e_2'\delta_1 \Downarrow_i v_a$ therefore from E-subExpE we know that $\exists i_2.e_1'[e_{b1}/x]\delta_1 \Downarrow_{i_2} v_a$

Instantiating with i_2 we get $e_2'[e_{b2}/x]\delta_2 \Downarrow v_b \wedge v_a \overset{s-i_1-i_2}{\approx}_{aV} v_b$

This proves the desired

15. $e = \Lambda.e'$:

    Similar reasoning as in the $\lambda y.e'$ case

16. $e = e'\,[]$:

    Similar reasoning as in the app case

17. $e = \mathsf{ret}\,e'$:

    We need to prove that $\mathsf{ret}\,e'\delta_1 \overset{s}{\approx}_{aE} \mathsf{ret}\,e'\delta_2$

    This means from Definition 43 it suffices to prove that

    $\forall i < s.\,\mathsf{ret}\,e'\delta_1 \Downarrow_i v_a \implies \mathsf{ret}\,e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

    This means that given some $i < s$ s.t $\mathsf{ret}\,e'\delta_1 \Downarrow_i v_a$ it suffices to prove that

    $\mathsf{ret}\,e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-R0)

    From E-val we know that $v_a = \mathsf{ret}\,e'\delta_1$ and $i = 0$

    From (FTE-R0) we need to prove that

    (a) $\mathsf{ret}\,e'\delta_2 \Downarrow v_b$:
        From E-val we know that $v_b = \mathsf{ret}\,e'\delta_2$

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:

We need to prove that
$\mathsf{ret}\, e'\delta_1 \overset{s}{\approx}_{aV} \mathsf{ret}\, e'\delta_2$

This means from Definition 43 it suffices to prove that
$\mathsf{ret}\, e'\delta_1 \Downarrow_i^k v_a \implies \mathsf{ret}\, e'\delta_2 \Downarrow^k v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This further means that given some $\mathsf{ret}\, e'\delta_1 \Downarrow_i^k v_a$ it suffices to prove that
$\mathsf{ret}\, e'\delta_2 \Downarrow^k v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-R1)

From E-return we know that $k = 0$ and $e'\delta_1 \Downarrow_i v_a$

<u>IH</u>: $e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$
This means from Definition 43 we have
$\forall j < s. e'\delta_1 \Downarrow_j v_a \implies e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-j}{\approx}_{aV} v_b$

Since we are given that $e'\delta_1 \Downarrow_i v_a$ therefore we get
$e'\delta_2 \Downarrow v_b \wedge v_a \overset{s-j}{\approx}_{aV} v_b$

Since $e'\delta_2 \Downarrow v_b$ therefore from E-return we also have
$\mathsf{ret}\, e'\delta_2 \Downarrow^0 v_b$
This proves the desired

18. $e = \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c$:

We need to prove that $\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \overset{s}{\approx}_{aE} \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2$

This means from Definition 43 it suffices to prove that

$\forall i < s.\, \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \Downarrow_i v_a \implies \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

This means that given some $i < s$ s.t $\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \Downarrow_i v_a$ it suffices to prove that

$\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$ (FTE-BI0)
From E-val we know that $v_a = \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1$ and $i = 0$

We need to prove

(a) $\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2 \Downarrow v_b$:
From E-val we know that $v_b = \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2$

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:
We need to prove that $\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \overset{s}{\approx}_{aV} \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2$

From Definition 43 it suffices to prove that
$\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \Downarrow_i^k v_{t1} \implies \mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2 \Downarrow^k v_{t2} \wedge v_{t1} \overset{s-i}{\approx}_{aV} v_{t2}$

This means that given $\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_1 \Downarrow_i^k v_{t1}$ it suffices to prove that
$\mathsf{bind}\, x = e_b \,\mathsf{in}\, e_c\delta_2 \Downarrow^k v_{t2} \wedge v_{t1} \overset{s-i}{\approx}_{aV} v_{t2}$ (F-BI1)

<u>IH1</u>: $e_b\delta_1 \overset{s}{\approx}_{aE} e_b\delta_2$

This means from Definition 43 we have

$\forall j < s.e_b\delta_1 \Downarrow_j v_{a1} \implies e_b\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$

Since we know that $\mathsf{bind}\, x = e_b \;\mathsf{in}\; e_c\delta_1 \Downarrow_i v_a$ therefore from E-bind we know that $\exists i_1.e_b\delta_1 \Downarrow_{i_1} v_{a1}$

Instantiating with i_1 we get $e_b\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$

Since v_{a1} is a mondic value and $v_{a1} \Downarrow^{k1}_{i'_1} v'_{a1}$

Since $v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$ therefore from Definition 43 we know that

$v_{a1} \Downarrow^{k1}_{i'_1} v'_{a1} \implies v_{b1} \Downarrow^{k1} v'_{b1} \wedge v'_{a1} \overset{s-i_1-i'_1}{\approx}_{aV} v'_{b1}$

Since we are given that $v_{a1} \Downarrow^{k1}_{i'_1} v'_{a1}$ therefore we have

$v_{b1} \Downarrow^{k1} v'_{b1} \wedge v'_{a1} \overset{s-i_1-i'_1}{\approx}_{aV} v'_{b1}$

<u>IH2</u>: $e_c[e'_{a1}/x]\delta_1 \overset{s-i_1-i'_1}{\approx}_{aE} e_c[e'_{b1}/x]\delta_2$

This means from Definition 43 we have

$\forall j < s.e_c[e'_{a1}/x]\delta_1 \Downarrow_j v_{a2} \implies e_c[e'_{b1}/x]\delta_2 \Downarrow v_b \wedge v_a \overset{s-i_1-i'_1-j}{\approx}_{aV} v_{b2}$

Since we know that $\mathsf{bind}\, x = e_b \;\mathsf{in}\; e_c\delta_1 \Downarrow_i v_a$ therefore from E-bind we know that $\exists i_2.e_c[e'_{a1}/x]\delta_1 \Downarrow_{i_2} v_{a2}$

Instantiating with i_2 we get $e_c[e'_{b1}/x]\delta_2 \Downarrow v_b \wedge v_{a2} \overset{s-i_1-i'_1-i_2}{\approx}_{aV} v_{b2}$

From E-bind we know that v_{a2} is a mondic value and $v_{a2} \Downarrow^{k2}_{i'_2} v'_{a2}$

Since $v_{a2} \overset{s-i_1-i'_1-i_2}{\approx}_{aV} v_{b2}$ therefore from Definition 43 we know that

$v_{a2} \Downarrow^{k2}_{i'_2} v'_{a2} \implies v_{b2} \Downarrow^{k2} v'_{b2} \wedge v'_{a2} \overset{s-i_1-i'_1-i_2-i'_2}{\approx}_{aV} v'_{b2}$

Since we are given that $v_{a2} \Downarrow^{k2}_{i'_2} v'_{a2}$ therefore we have

$v_{b2} \Downarrow^{k2} v'_{b2} \wedge v'_{a2} \overset{s-i_1-i'_1-i_2-i'_2}{\approx}_{aV} v'_{b2}$

This proves the desired

19. $e = \uparrow^n$:

    Trivial

20. $e = \mathsf{release}\, e_r = x \;\mathsf{in}\; e_c$:

    Similar reasoning as in the bind case

21. $e = \mathsf{store}\, e$:

    Similar reasoning as in the return case

□

Lemma 48 (Equivalence relation of λ-amor is reflexive for values)**.** $\forall v, s.\ v \overset{s}{\approx}_{aV} v$

Proof. Instantiating Theorem 47 with . for δ_1 and δ_2, v for e and with the given s we get $v \overset{s}{\approx}_{aE} v$

From Definition 43 this means we have

$\forall i < s.v \Downarrow_i v_a \implies v \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$

Instantiating it with i as 0 and since we knwo that $v \Downarrow_0 v$ therefore we get the desired ☐

Lemma 49 (Property of app rule in λ-Amor)**.** $\forall e_1, e_2, e, s.$

$e_1 \overset{s}{\approx}_{aE} e_2 \implies e\ e_1 \overset{s}{\approx}_{aE} e\ e_2$

Proof. We get the desired from Theorem 47 ☐

Lemma 50 (Lemma for app1 : empty stack)**.** $\forall t, u, \rho, \theta, v_a, v_1, j.$

$\Theta; \Delta; . \vdash_{-} (\!|(t\ u, \rho, \epsilon)|\!) : - \ \wedge$

$\overline{\Theta; \Delta; .} \vdash_{-} (\!|(t, \rho, (u, \rho).\epsilon)|\!) : - \ \wedge$

$\overline{(\!|(t\ u, \rho, \epsilon)|\!)}() \Downarrow v_a \Downarrow^j v_1 \implies$

$\exists v_b, v_2.\ \overline{(\!|(t, \rho, (u, \rho).\epsilon)|\!)}() \Downarrow v_b \Downarrow^j v_2 \wedge \forall s.v_1 \overset{s}{\approx}_{aE} v_2$

Proof. From Definition 40 know that

$(\!|(t\ u, \rho, \epsilon)|\!) = (\!|t\ u, \rho|\!) =$

$(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_n|\!) \qquad$ (A1.0)

Similarly from Definition 40 we also have

$(\!|(t, \rho, (u, \rho).\epsilon)|\!) = (\!|(\!|(t, \rho)|\!)\ (\!|(u, \rho)|\!), ., \epsilon|\!) = (\!|(\!|(t, \rho)|\!)\ (\!|(u, \rho)|\!), .|\!) = (\!|(t, \rho)|\!)\ (\!|(u, \rho)|\!) =$

$((\lambda x_1 \ldots x_n.t)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_n|\!))\ \ (\lambda x_1 \ldots x_n.u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_n|\!) \qquad$ (A1.1)

Since $\overline{\Theta; \Delta; .} \vdash_{-} (\!|(t\ u, \rho, \epsilon)|\!) : -$ therefore from Theorem 22 we know that

$\overline{(\!|(t\ u, \rho, \epsilon)|\!)} =$

$\overline{(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_n|\!)} =$

$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,n}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$

where

$E_c = b\ (coerce1\ !e_{t2,n}\ c)\ d$

$e_{t1,n} = \overline{(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_{n-1}|\!)}$

$e_{t2,n} = \overline{(\!|\mathsf{C}_n|\!)}$

$\overline{e_{t1,n}} =$

$\overline{(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_{n-1}|\!)} =$

$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,n-1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$

where

$E_c = b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$e_{t1,n-1} = \overline{(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_{n-2}|\!)}$

$e_{t2,n-1} = \overline{(\!|\mathsf{C}_{n-1}|\!)}$

...

$\overline{e_{t1,2}} =$

$\overline{(\lambda x_1 \ldots x_n.t\ u)\ (\!|\mathsf{C}_1|\!)} =$

$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$

where

$E_c = b\ (coerce1\ !e_{t2,1}\ c)\ d$

$e_{t1,1} = \overline{(\lambda x_1 \ldots x_n.t\ u)}$

$e_{t2,1} = (\!|\mathtt{C}_1|\!)$

$e_{t1,1} =$
$\overline{(\lambda x_1 \ldots x_n.t\ u)} =$
$\lambda p_1.\,\mathsf{ret}\ \lambda y.\lambda p_2.\,\mathsf{let}\,!\,x = y\ \mathsf{in}\ \mathsf{release} - = p_1\ \mathsf{in}\ \mathsf{release} - = p_2\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ e_{t2}\ a$
where
$e_{t2} = \overline{(\lambda x_2 \ldots x_n.t\ u)}$

$\ldots$

$e_{tn-1} =$
$\overline{(\lambda x_{n-1} x_n.t\ u)} =$
$\lambda p_1.\,\mathsf{ret}\ \lambda y.\lambda p_2.\,\mathsf{let}\,!\,x = y\ \mathsf{in}\ \mathsf{release} - = p_1\ \mathsf{in}\ \mathsf{release} - = p_2\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ e_{tn}\ a$
where
$e_{tn} = \overline{(\lambda x_n.t\ u)}$

$e_{tn} =$
$\overline{(\lambda x_n.t\ u)} =$
$\lambda p_1.\,\mathsf{ret}\ \lambda y.\lambda p_2.\,\mathsf{let}\,!\,x = y\ \mathsf{in}\ \mathsf{release} - = p_1\ \mathsf{in}\ \mathsf{release} - = p_2\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ e'_t\ a$
where
$e'_t = \overline{(t\ u)}$

$e'_t =$
$\overline{(t\ u)} =$
$\lambda p.\mathsf{release} - = p\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ \mathsf{bind}\, b = e_t\ a\ \mathsf{in}\ \mathsf{bind}\, c = \mathsf{store!}()\ \mathsf{in}\ \mathsf{bind}\, d = \mathsf{store}()\ \mathsf{in}\ E_c$
where
$E_c = b\ (coerce1\ !e_u\ c)\ d$
$e_t = \overline{t}$
$e_u = \overline{u}$

Since we know that $\overline{(\!|(t\ u, \rho, \epsilon)|\!)}() \Downarrow v_a \Downarrow^j \overline{v_1}$ therefore from the reduction rule we know that $\exists j_l, L.\overline{(\!|t|\!)}\ () \Downarrow - \Downarrow^{j_l} L$ and $\exists j_a.L\ (coerce1!\overline{(\!|u|\!)}!())\ () \Downarrow - \Downarrow^{j_l} v_1$ s.t $j = j_l + j_a$

Similarly from (A1.1) we know that
$(\!|(t, \rho, (u, \rho).\epsilon)|\!) =$
$((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))\ \ (\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!)$

Since $\Theta; \Delta; . \vdash_{-} (\!|(t, \rho, (u, \rho).\epsilon)|\!) : -$ therefore from Theorem 22 we know that
$\overline{(\!|(t, \rho, (u, \rho).\epsilon)|\!)} =$
$\overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))\ \ ((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))} =$
$\lambda p.\mathsf{release} - = p\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ \mathsf{bind}\, b = e_{t,n}\ a\ \mathsf{in}\ \mathsf{bind}\, c = \mathsf{store!}()\ \mathsf{in}\ \mathsf{bind}\, d = \mathsf{store}()\ \mathsf{in}\ E_c$
where
$E_c = b\ (coerce1\ !e_{u,n}\ c)\ d$
$e_{t,n} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))}$
$e_{u,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))}$

$e_{t,n} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))} =$
$\lambda p.\mathsf{release} - = p\ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ \mathsf{bind}\, b = e_{t1,n}\ a\ \mathsf{in}\ \mathsf{bind}\, c = \mathsf{store!}()\ \mathsf{in}\ \mathsf{bind}\, d = \mathsf{store}()\ \mathsf{in}\ E_c$

where
$E_c = b\ (coerce1\ !e_{t2,n}\ c)\ d$
$e_{t1,n} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-1}|\!))}$
$e_{t2,n} = \overline{\mathtt{C}_n}$

$e_{t1,n} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-1}|\!))} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,n-1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2,n-1}\ c)\ d$
$e_{t1,n-1} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-2}|\!))}$
$e_{t2,n-1} = \overline{\mathtt{C}_{n-1}}$

...

$e_{t1,2} = \overline{((\lambda x_1 \ldots x_n.t)\ (\!|\mathtt{C}_1|\!))} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{l1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2,1}\ c)\ d$
$e_{l1} = \overline{(\lambda x_1 \ldots x_n.t)}$
$e_{t2,1} = \overline{\mathtt{C}_1}$

$e_{l1} = \overline{(\lambda x_1 \ldots x_n.t)} =$
$\lambda p_1.\,\mathsf{ret}\ \lambda y.\lambda p_2.\,\mathsf{let}\,!\,x_1 = y \mathsf{\ in\ release} - = p_1 \mathsf{\ in\ release} - = p_2 \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ } e_{l2}\ a$
where
$e_{l2} = \overline{(\lambda x_2 \ldots x_n.t)}$

...

$e_{ln} = \overline{(\lambda x_n.t)} =$
$\lambda p_1.\,\mathsf{ret}\ \lambda y.\lambda p_2.\,\mathsf{let}\,!\,x_n = y \mathsf{\ in\ release} - = p_1 \mathsf{\ in\ release} - = p_2 \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ } e_T\ a$
where
$e_T = \overline{t}$ (A1.2)

Similarly we also have
$e_{u,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))}$

$e_{u,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_n|\!))} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{u1,n}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{u2,n}\ c)\ d$
$e_{u1,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-1}|\!))}$
$e_{u2,n} = \overline{\mathtt{C}_n}$

$e_{u1,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-1}|\!))} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{u1,n-1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$e_{u1,n-1} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-2}|\!))}$

$e_{u2,n-1} = \overline{\mathtt{C}_{n-1}}$

$e_{u1,n-1} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-2}|\!))} =$
$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n-2}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c$
where
$E_c = b\ (coerce1\ !e_{u2,n-2}\ c)\ d$
$e_{u1,n-2} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)\ \ldots\ (\!|\mathtt{C}_{n-3}|\!))}$
$e_{u2,n-2} = \overline{\mathtt{C}_{n-2}}$

...

$e_{u1,2} = \overline{(\lambda x_1 \ldots x_n.u)\ (\!|\mathtt{C}_1|\!)} =$
$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c$
where
$E_c = b\ (coerce1\ !e_{u2,1}\ c)\ d$
$e_{u1,1} = \overline{(\lambda x_1 \ldots x_n.u)}$
$e_{u2,1} = \overline{\mathtt{C}_1}$

$e_{u1,1} = \overline{(\lambda x_1 \ldots x_n.u)} =$
$\lambda p_1.\,\mathsf{ret}\,\lambda y.\lambda p_2.\,\mathsf{let}\,!\,x_1 = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,1}\ a$
where
$e_{U,1} = \overline{(\lambda x_2 \ldots x_n.u)}$

$e_{U,1} = \overline{(\lambda x_2 \ldots x_n.u)} =$
$\lambda p_1.\,\mathsf{ret}\,\lambda y.\lambda p_2.\,\mathsf{let}\,!\,x_2 = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,2}\ a$
where
$e_{U,2} = \overline{(\lambda x_3 \ldots x_n.u)}$
...

$e_{U,n-1} = \overline{(\lambda x_n.u)} =$
$\lambda p_1.\,\mathsf{ret}\,\lambda y.\lambda p_2.\,\mathsf{let}\,!\,x_n = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,n}\ a$
where
$e_{U,n} = \overline{u}$ (A1.3)

$E_0 = \lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_0$
$E'_0 = b\ (coerce1\ !e_{u,n}\ c)\ d$
$v_b = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_0$
$E_{0.1} = \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u,n}\ c)\ d$
$E_{0.2} = \mathsf{bind}\, b = e_{t,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u,n}\ c)\ d$
$E_{0.3} = \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u,n}\ c)\ d$
$E_{0.4} = \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u,n}\ c)\ d$
$e_{t,n} = \lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{t,n}$
$E'_{t,n} = b\ (coerce1\ !e_{t2,n}\ c)\ d$
$E_{t,n,1} = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{t,n}$
$E_{t,n,1.1} = \mathsf{bind}\, b = e_{t1,n}\ () \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{t,n}$

$e_{t1,n} = \lambda p.$
$\textsf{release} - = p \textsf{ in bind } a = \textsf{store}() \textsf{ in bind } b = e_{t1,n-1}\ a \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } E'_{t1,n}$

$E'_{t1,n} = b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$E_{t1,n,1} = \textsf{release} - = () \textsf{ in bind } a = \textsf{store}() \textsf{ in bind } b = e_{t1,n-1}\ a \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$E_{t1,n,2} = \textsf{bind } b = e_{t1,n-1}\ () \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$E_{t1,n,3} = \textsf{bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$E_{t1,n,4} = \textsf{bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,n-1}\ c)\ d$

$e_{t1,2} = \lambda p.$
$\textsf{release} - = p \textsf{ in bind } a = \textsf{store}() \textsf{ in bind } b = e_{l1}\ a \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } E'_{t1,2}$

$E'_{t1,2} = b\ (coerce1\ !e_{t2,1}\ c)\ d$

$E_{t1,2,1} = \textsf{release} - = () \textsf{ in bind } a = \textsf{store}() \textsf{ in bind } b = e_{l1}\ a \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,1}\ c)\ d$

$E_{t1,2,2} = \textsf{bind } b = e_{l1}\ a \textsf{ in bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,1}\ c)\ d$

$E_{t1,2,3} = \textsf{bind } c = \textsf{store!}() \textsf{ in bind } d = \textsf{store}() \textsf{ in } b\ (coerce1\ !e_{t2,1}\ c)\ d$

$e_{l1} = \lambda p_1.\, \textsf{ret } \lambda y.\lambda p_2.\, \textsf{let } !\, x_1 = y \textsf{ in release} - = p_1 \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l2}\ a$

$E_{l1} = \textsf{ret } \lambda y.\lambda p_2.\, \textsf{let } !\, x_1 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l2}\ a$

$E_{l,1,1} = \lambda y.\lambda p_2.\, \textsf{let } !\, x_1 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l2}\ a$

$E_{l,1,2} = \textsf{let } !\, x_1 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l2}\ a$

$E_{l,1,3} = \textsf{release} - = () \textsf{ in release} - = () \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l2}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$

$E_{l2} = \textsf{ret } \lambda y.\lambda p_2.\, \textsf{let } !\, x_2 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l3}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$

$E_{l,2,1} = \lambda y.\lambda p_2.\, \textsf{let } !\, x_2 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l3}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$

$E_{l,2,2} = \textsf{release} - = () \textsf{ in release} - = () \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l3}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2]$

$E_{l3} = \textsf{ret } \lambda y.\lambda p_2.\, \textsf{let } !\, x_3 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l4}\ a [(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2]$

$E_{l,3,1} = \lambda y.\lambda p_2.\, \textsf{let } !\, x_3 = y \textsf{ in release} - = () \textsf{ in release} - = p_2 \textsf{ in bind } a = \textsf{store}() \textsf{ in } e_{l4}\ a [(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2]$

$D_{n-3}2$:

$$\frac{\vdots}{E_{l,n-3,1}\ (coerce1\ !\overline{(\!|\mathtt{C}_{n-3}|\!)}\ !())\ ()\ \Downarrow^{-} E_{l,n-2,1}}$$

$D_1 2$:

$$\frac{}{E_{l,1,1}(coerce1!\overline{(\!|\mathtt{C}_1|\!)}!())() \Downarrow - \Downarrow E_{l,2,1}}$$

$D_1 1$:

$$\frac{}{E_{l1} \Downarrow^{0} E_{l,1,1}}$$

$D_2 2.3$:

$$\frac{}{E_{l3} \Downarrow^{-} E_{l,3,1}}$$

$D_2 2.2$:

$$\frac{}{e_{l3}\ ()[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2] \Downarrow E_{l3}}$$

$D_2 2.1$:

$$\frac{}{(coerce1\ !\overline{(\!|\mathtt{C}_2|\!)}\ !()) \Downarrow !(\overline{(\!|\mathtt{C}_2|\!)}\ ())}$$

$D_2 2$:

$$\frac{\dfrac{D_2 2.1}{E_{l,2,1}[(coerce1\ !\overline{(\!|\mathtt{C}_2|\!)}\ !())/y][()/p_2] \Downarrow E_{l1,2,2}} \qquad D_2 2.2 \qquad D_2 2.3}{E_{l,2,1}\ (coerce1\ !e_{t2,1}\ !())\ ()\ \Downarrow^{-}\ E_{l,3,1}}$$

$D_2 1$:

$$\frac{\dfrac{}{e_{l1}\ () \Downarrow E_{l1}} \qquad D_1 1 \qquad D_1 2}{E_{t1,2,1} \Downarrow^{0} E_{l,2,1}}$$

$D_3 2$:

$$\frac{\vdots}{E_{l,3,1}\ (coerce1\ !\overline{(\!|\mathtt{C}_3|\!)}\ !())\ ()\ \Downarrow^{-}\ E_{l,4,1}}$$

$D_3 1$:

$$\frac{e_{t1,2}\ () \Downarrow E_{t1,2,1} \qquad D_2 1 \qquad D_2 2}{E_{t1,3,1} \Downarrow E_{l,3,1}}$$

$D_{n-2} 2$:

$$\frac{\vdots}{E_{l,(n-2),1}\ (coerce1\ !\overline{(\!|\mathtt{C}_{n-2}|\!)}\ !())\ ()\ \Downarrow^{0}\ E_{l,(n-1),1}}$$

$D_{n-2} 1$:

$$\frac{\dfrac{}{e_{t1,n-3}\ () \Downarrow E_{t1,n-3,1}} \qquad \dfrac{\dfrac{}{e_{t1,3}() \Downarrow E_{t1,3,1}} \qquad D_3 1 \qquad D_3 2}{\vdots} \qquad D_{n-3} 2}{E_{t1,n-2,1} \Downarrow^{-} E_{l,n-2,1}}$$

$D_{n-1} 2$:

$$\frac{\vdots}{E_{l,n-1,1}(coerce1!\overline{(\!|\mathtt{C}_{n-1}|\!)}!())() \Downarrow^{0} E_{l,n,1}}$$

$D_{n-1} 1$:

$$\frac{\dfrac{}{e_{t1,n-2}\ () \Downarrow E_{t1,n-2,1}} \qquad D_{n-2} 1 \qquad D_{n-2} 2}{E_{t1,n-1,1} \Downarrow E_{l,n-1,1}}$$

$D_n 2$:

$$\frac{\dfrac{\dfrac{}{\overline{t}[!(\overline{(\!|\mathtt{C}_n|\!)}\ ())/x_n] \Downarrow - \Downarrow^{j_l} L}\ \text{By inversion}}{E_{l,n,1}[(coerce1!\overline{(\!|\mathtt{C}_n|\!)}!())/x_n][()/p_2] \Downarrow^{j_l} L}}{E_{l,n,1}(coerce1!\overline{(\!|\mathtt{C}_n|\!)}!())\ () \Downarrow^{j_l} L}$$

$D_n 1$:

$$\frac{\dfrac{}{e_{t1,n-1}\ () \Downarrow E_{t1,n-1,1}} \qquad D_{(n-1)} 1 \qquad D_{(n-1)} 2}{E_{t1,n,1} \Downarrow^{j} E_{l,n,1}}$$

D2:

$$\frac{\dfrac{}{v_a \Downarrow^{j} v_1}\ \text{Given} \qquad \dfrac{}{v_a \overset{s}{\approx}_{aV} v_b}}{v_b \Downarrow^{j} v_2 \qquad v_1 \overset{s}{\approx}_{aV} v_2}\ \text{Definition 43}$$

T1:

$$\dfrac{\dfrac{\dfrac{}{L\ (coerce1\ !e_{u,n}\ !())\ () \Downarrow - \Downarrow^{j_a} v_b \qquad v_a \overset{s}{\approx}_{aV} v_b}\text{Claim, Lemma 49, Definition 43}}{E_{0.4}[L/b][!()/c] \Downarrow^{j_a} v_b}}{E_{0.3}[L/b] \Downarrow^{j_a} v_b}$$

T0:

$$\dfrac{\dfrac{}{e_{t1,n}\ () \Downarrow E_{t1,n,1}} \qquad D_n1 \qquad D_n2}{E_{t,n,1} \Downarrow^{j} L}\text{E-bind}$$

D0.0:

$$\dfrac{\dfrac{\dfrac{}{\mathsf{store}() \Downarrow^{0} ()} \qquad \dfrac{\dfrac{}{e_{t,n}\ () \Downarrow E_{t,n,1}} \qquad T0 \qquad T1 \qquad D2}{E_{0.2} \Downarrow^{j} v_2}\text{E-bind}}{E_{0.1} \Downarrow^{j} v_2}\text{E-bind}}{v_b \Downarrow^{j} v_2}\text{E-release}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{E_0() \Downarrow v_b} \qquad D0.0}{E_0\ () \Downarrow v_b \Downarrow^{j} v_2}}{((\lambda x_1 \ldots x_n.t)\ \llparenthesis \mathtt{C}_1 \rrparenthesis\ \ldots\ \llparenthesis \mathtt{C}_n \rrparenthesis)\ (\lambda x_1 \ldots x_n.u)\ \llparenthesis \mathtt{C}_1 \rrparenthesis\ \ldots\ \llparenthesis \mathtt{C}_n \rrparenthesis\ () \Downarrow v_b \Downarrow^{j} v_2}}{\llparenthesis (t, \rho, (u, \rho).\epsilon) \rrparenthesis\ () \Downarrow v_b \Downarrow^{j} v_2}$$

Claim: $\forall s. coerce1\ !\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ !() \overset{s}{\approx}_{aE} coerce1\ !e_{u,n}\ !()$

Proof

From Definition 43 it suffices to prove
$\forall i < s. coerce1\ !\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ !() \Downarrow_i v_1 \implies coerce1\ !e_{u,n}\ !() \Downarrow v_2 \wedge v_1 \overset{s-i}{\approx}_{aV} v_2$

This further means that given some $i < s$ s.t $coerce1\ !\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ !() \Downarrow_i v_1$ and we need to prove
$coerce1\ !e_{u,n}\ !() \Downarrow v_2 \wedge v_1 \overset{s-i}{\approx}_{aV} v_2 \qquad \text{(C0)}$

Since we are given that $coerce1\ !\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ !() \Downarrow v_1$
This means from Definition 31 we have $v_1 = !(\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ ())$

Similarly again from Definition 31 we know that
$v_2 = !(e_{u,n}\ ())$

In order to prove that $!(\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ ()) \overset{s-i}{\approx}_{aE} !(e_{u,n}\ ())$
from Definition 43 it suffices to prove that
$(\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ ()) \overset{s-i}{\approx}_{aE} (e_{u,n}\ ())$

Using Definition 43 it suffices to prove
$\forall j < (s-i). (\overline{u}[\overline{\llparenthesis \mathtt{C}_1 \rrparenthesis}\ ()/x_1] \ldots [\overline{\llparenthesis \mathtt{C}_n \rrparenthesis}\ ()/x_n]\ ()) \Downarrow_j v_1' \implies (e_{u,n}\ ()) \Downarrow v_2' \wedge v_1' \overset{s-i-j}{\approx}_{aV} v_2'$

This means given some $j < (s-i)$ s.t $(\overline{u}[\overline{(\!|\mathsf{C}_1|\!)}\ ()/x_1]\ldots[\overline{(\!|\mathsf{C}_n|\!)}\ ()/x_n]\ ()) \Downarrow_j v'_1$
it suffices to prove that
$(e_{u,n}\ ()) \Downarrow v'_2 \wedge v'_1 \stackrel{s-i-j}{\approx}_{aV} v'_2$

From the embedding of dlPCF into λ-amor we know that v'_1 is a value of monadic type

Since we know that
$e_{u,n} = \lambda p.$
$\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{u,n}$
where
$E'_{u,n} = b\ (coerce1\ !e_{u2,n}\ c)\ d$
$e_{u1,n} = \overline{((\lambda x_1 \ldots x_n.u)\ (\!|\mathsf{C}_1|\!)\ \ldots\ (\!|\mathsf{C}_{n-1}|\!))}$
$e_{u2,n} = \overline{\mathsf{C}_n}$

$e_{u,n}\ () \Downarrow v'_2$ from E-app where
$v'_2 = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n}\ c)\ d$

Now we need to prove that $v'_1 \stackrel{s-i-j}{\approx}_{aV} v'_2$

From Definition 43 it suffices to prove that
$v'_1 \Downarrow^k_l v'_a \implies v'_2 \Downarrow^k v'_b \wedge v'_a \stackrel{s-i-j-l}{\approx}_{aV} v'_b$
This means given $v'_1 \Downarrow^k_l v'_a$ it suffices to prove
$v'_2 \Downarrow^k v'_b \wedge v'_a \stackrel{s-i-j-l}{\approx}_{aV} v'_b$

$v'_2 = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n}\ c)\ d$
$E_{u,n,1} = \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n}\ c)\ d$
$E_{u,n,1.1} = \mathsf{bind}\, b = e_{u1,n}\ () \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n}\ c)\ d$
$E_{u,n,1.2} = \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n}\ c)\ d$
$e_{u1,n} = \lambda p.$
$\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n-1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{u1,n}$
$E'_{u1,n} = b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$E_{u1,n,1} = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{u1,n-1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$E_{u1,n,2} = \mathsf{bind}\, b = e_{u1,n-1}\ () \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$E_{u1,n,3} = \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$E_{u1,n,4} = \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,n-1}\ c)\ d$
$e_{u1,2} = \lambda p.$
$\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{l1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E'_{u1,2}$
$E'_{u1,2} = b\ (coerce1\ !e_{u2,1}\ c)\ d$
$E_{u1,2,1} = \mathsf{release} - = () \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{l1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,1}\ c)\ d$
$E_{u1,2,2} = \mathsf{bind}\, b = e_{l1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,1}\ c)\ d$
$E_{u1,2,3} = \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } b\ (coerce1\ !e_{u2,1}\ c)\ d$
$e_{l1} = \lambda p_1.\, \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\, !\, x_1 = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,2}\ a$
$E_{l1} = \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\, !\, x_1 = y \text{ in } \mathsf{release} - = () \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,2}\ a$
$E_{l,1,1} = \lambda y.\lambda p_2.\, \mathsf{let}\, !\, x_1 = y \text{ in } \mathsf{release} - = () \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,2}\ a$
$E_{l,1,2} = \mathsf{let}\, !\, x_1 = y \text{ in } \mathsf{release} - = () \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{U,2}\ a$

$E_{l,1,3} = \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,2}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$
$E_{l2} = \mathsf{ret}\,\lambda y.\lambda p_2.\, \mathsf{let}\,!\,x_2 = y \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = p_2 \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,3}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$
$E_{l,2,1} = \lambda y.\lambda p_2.\, \mathsf{let}\,!\,x_2 = y \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = p_2 \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,3}\ a[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]$
$E_{l,2,2} = (\mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,3}\ a)\ S_2$
$E_{l3} = (\mathsf{ret}\,\lambda y.\lambda p_2.\, \mathsf{let}\,!\,x_3 = y \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = p_2 \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,4}\ a)\ S_2$
$E_{l,3,1} = (\lambda y.\lambda p_2.\, \mathsf{let}\,!\,x_3 = y \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = p_2 \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,4}\ a)\ S_2$
$S_2 = [(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2]$
$E_{l,n,1} = (\lambda y.\lambda p_2.\, \mathsf{let}\,!\,x_n = y \mathsf{~in~} \mathsf{release} - = () \mathsf{~in~} \mathsf{release} - = p_2 \mathsf{~in~} \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} e_{U,n}\ a)\ S_{n-1}$
$S_{n-1} = [(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]\ldots[(\overline{(\!|\mathtt{C}_{n-1}|\!)}\ ())/x_{n-1}]$

$D_{n-3}2$:

$$\frac{\vdots}{E_{l,(n-3),1}\ (coerce1\ !\overline{(\!|\mathtt{C}_{n-3}|\!)}\ !())\ ()\Downarrow^0 E_{l,(n-2),1}}$$

D_12:

$$\frac{}{E_{l,1,1}(coerce1!\overline{(\!|\mathtt{C}_1|\!)}!())()\Downarrow - \Downarrow E_{l,2,1}}$$

D_11:

$$\frac{}{E_{l1}\Downarrow^0 E_{l,1,1}}$$

$D_22.3$:

$$\frac{}{E_{l3}\Downarrow^0 E_{l,3,1}}$$

$D_22.2$:

$$\frac{}{e_{l3}\ ()[(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1][(\overline{(\!|\mathtt{C}_2|\!)}\ ())/x_2]\Downarrow E_{l3}}$$

$D_22.1$:

$$\frac{}{(coerce1\ !\overline{(\!|\mathtt{C}_2|\!)}\ !())\Downarrow !(\overline{(\!|\mathtt{C}_2|\!)}\ ())}$$

D_22:

$$\frac{\dfrac{D_22.1}{E_{l,2,1}[(coerce1\ !\overline{(\!|\mathtt{C}_2|\!)}\ !())/y][()/p_2]\Downarrow E_{l1,2,2}}\qquad D_22.2\qquad D_22.3}{E_{l,2,1}\ (coerce1\ !e_{u2,1}\ !())\ ()\Downarrow^0 E_{l,3,1}}$$

D_21:

$$\frac{\dfrac{}{e_{l1}\ ()\Downarrow E_{l1}}\qquad D_11\qquad D_12}{E_{u1,2,1}\Downarrow^0 E_{l,2,1}}$$

D_32:

$$\frac{\vdots}{E_{l,3,1}\ (coerce1\ !\overline{(\!|\mathtt{C}_3|\!)}\ !())\ ()\Downarrow^0 E_{l,4,1}}$$

D_31:

$$\frac{e_{u1,2}\ ()\Downarrow E_{u1,2,1}\qquad D_21\qquad D_22}{E_{u1,3,1}\Downarrow E_{l,3,1}}$$

$D_{n-2}2$:

$$\frac{\vdots}{E_{l,(n-2),1}\ (coerce1\ !\overline{(\!|\mathtt{C}_{n-2}|\!)}\ !())\ ()\ \Downarrow^0 E_{l,(n-1),1}}$$

$D_{n-2}1$:

$$\frac{\overline{e_{u1,n-3}\ ()\Downarrow E_{u1,n-3,1}} \quad \dfrac{\dfrac{\overline{e_{u1,3}()\Downarrow E_{u1,3,1}} \quad D_3 1 \quad D_3 2}{\vdots} \quad D_{n-3}2}{}}{E_{u1,n-2,1}\Downarrow^0 E_{l,n-2,1}}$$

$D_{n-1}2$:

$$\frac{\vdots}{E_{l,n-1,1}(coerce1!\overline{(\!|\mathtt{C}_{n-1}|\!)}!())()\Downarrow^0 E_{l,n,1}}$$

$D_{n-1}1$:

$$\frac{\overline{e_{u1,n-2}\ ()\Downarrow E_{u1,n-2,1}} \quad D_{n-2}1 \quad D_{n-2}2}{E_{u1,n-1,1}\Downarrow^0 E_{l,n-1,1}}$$

$D_n 2$:

$$\frac{\dfrac{\overline{\overline{u}[!(\overline{(\!|\mathtt{C}_1|\!)}\ ())/x_1]\ldots[!(\overline{(\!|\mathtt{C}_n|\!)}\ ())/x_n]\Downarrow v_1'\Downarrow^k v_a'}\ \text{Given}}{E_{l,n,1}[(coerce1!\overline{(\!|\mathtt{C}_n|\!)}!())/x_n][()/p_2]\Downarrow v_1'\Downarrow^k v_a'}}{E_{l,n,1}(coerce1!\overline{(\!|\mathtt{C}_n|\!)}!())\ ()\Downarrow^k v_a'}$$

$D_n 1$:

$$\frac{\overline{e_{u1,n-1}\ ()\Downarrow E_{u1,n-1,1}} \quad D_{(n-1)}1 \quad D_{(n-1)}2}{E_{u1,n,1}\Downarrow^0 E_{l,n,1}}$$

Main derivation:

$$\frac{\dfrac{\overline{e_{u1,n}\ ()\Downarrow E_{u1,n,1}} \quad D_n 1 \quad D_n 2}{E_{u,n,1}\Downarrow^k v_a'}\ \text{E-bind}}{v_2'\Downarrow^k v_a'}\ \text{E-release}$$

From Lemma 48 we get $v_a' \overset{s-i-j-l}{\approx}_{aV} v_a'$

$\square$

Lemma 51 (Lemma for app1: non-empty stack). $\forall t,u,\rho,\theta,v'_{\epsilon1},v_{\epsilon1},v'_{\epsilon2},v_{\epsilon2},v_{\theta1},j,j',j''$.
$(t\ u,\rho,\epsilon)$ and $(t,\rho,(u,\rho).\epsilon)$ are well-typed
$(t\ u,\rho,\theta)$ and $(t,\rho,(u,\rho).\theta)$ are well-typed
$(t\ u,\rho,\epsilon)\to(t,\rho,(u,\rho).\epsilon)\ \wedge\ (t\ u,\rho,\theta)\to(t,\rho,(u,\rho).\theta)\ \wedge$
$\overline{(\!|(t\ u,\rho,\epsilon)|\!)}\ ()\Downarrow v'_{\epsilon1}\Downarrow^j v_{\epsilon1}\ \wedge\ \overline{(\!|(t,\rho,(u,\rho).\epsilon)|\!)}\ ()\Downarrow v'_{\epsilon2}\Downarrow^{j'} v_{\epsilon2}\ \wedge\ \forall s.v_{\epsilon1}\overset{s}{\approx}_{aV} v_{\epsilon2}\ \wedge$
$\overline{(\!|(t\ u,\rho,\theta)|\!)}\ ()\Downarrow v'_{\theta1}\Downarrow^{j''} v_{\theta1}$
$\Longrightarrow$
$\exists v'_{\theta2},v_{\theta2},j'''.\ \overline{(\!|(t,\rho,(u,\rho).\theta)|\!)}\ ()\Downarrow v'_{\theta2}\Downarrow^{j'''} v_{\theta2}\ \wedge\ (j-j')=(j''-j''')\ \wedge\ \forall s.v_{\theta1}\overset{s}{\approx}_{aV} v_{\theta2}$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:
   Directly from given

2. Case $\theta = \mathsf{C}'.\theta'$:
   Let $\theta' = \mathsf{C}'_1 \dots \mathsf{C}'_n$ and $\theta'' = \mathsf{C}'_1 \dots \mathsf{C}'_{n-1}$

   Given:
   $(t\ u, \rho, \mathsf{C}'.\theta')$ and $(t, \rho, (u, \rho).\mathsf{C}'.\theta')$ are well-typed $\wedge$
   $(t\ u, \rho, \mathsf{C}'.\theta') \rightarrow (t, \rho, (u, \rho).\mathsf{C}'.\theta') \wedge \overline{(\!|(t\ u, \rho, \mathsf{C}'.\theta')|\!)}\ () \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$
   We need to prove that
   $\exists v'_{\theta 2}, v_{\theta 2}, j'''.$
   $$\overline{(\!|(t, \rho, (u, \rho).\mathsf{C}'.\theta')|\!)}\ () \Downarrow v'_{\theta 2} \Downarrow^{j'''} v_{\theta 2} \wedge (j - j') = (j'' - j''') \wedge \forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \qquad \text{(ET-0)}$$

   From IH we know
   $(t\ u, \rho, \mathsf{C}'.\theta'')$ and $(t, \rho, (u, \rho).\mathsf{C}'.\theta'')$ are well-typed $\wedge$
   $(t\ u, \rho, \mathsf{C}'.\theta'') \rightarrow (t, \rho, (u, \rho).\mathsf{C}'.\theta'') \wedge \overline{(\!|(t\ u, \rho, \mathsf{C}'.\theta'')|\!)}\ () \Downarrow v'_{\theta 11} \Downarrow^{j''_1} v_{\theta 11} \implies \exists j'''_1, v'_{\theta 22}, v_{\theta 22}.$
   $$\overline{(\!|(t, \rho, (u, \rho).\mathsf{C}'.\theta'')|\!)}\ () \Downarrow v'_{\theta 22} \Downarrow^{j'''_1} v_{\theta 22} \wedge (j - j') = (j''_1 - j'''_1) \wedge \forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \qquad \text{(ET-IH)}$$

   From Definition 39 and Definition 40 we know that
   $$(\!|(t\ u, \rho, \mathsf{C}'.\theta')|\!) = (\!|((\!|t\ u, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))|\!) \qquad \text{(ET-1)}$$

   Since $(t\ u, \rho, \mathsf{C}'.\theta')$ is well typed therefore we know that
   $\overline{(\!|(t\ u, \rho, \mathsf{C}'.\theta')|\!)} = \overline{(\!|((\!|t\ u, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))|\!)} =$
   $\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
   where
   $E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
   $e_{t1} = \overline{(\!|((\!|t\ u, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!))|\!)}$
   $e_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ (ET-1.1)

   From Krivine reduction (app rule) we also know that $(t\ u, \rho, \mathsf{C}'.\theta'') \rightarrow (t, \rho, (u, \rho).\mathsf{C}'.\theta'')$
   Also since we know that $\overline{(\!|(t\ u, \rho, \mathsf{C}'.\theta')|\!)}\ () \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$ therefore we also know that $\exists j''_1, v'_1, v_1. e_{t1}() \Downarrow v_1 \Downarrow^{j''_1} v'_1$

   Also since we know that
   $(t\ u, \rho, \mathsf{C}'.\theta')$ and $(t, \rho, (u, \rho).\mathsf{C}'.\theta')$ are well-typed
   therefore from Lemma 56 we also know that
   $(t\ u, \rho, \mathsf{C}'.\theta'')$ and $(t, \rho, (u, \rho).\mathsf{C}'.\theta'')$ are well-typed

   Therefore from (ET-IH) we have

$\exists j_1''', v'_{\theta 22}, v_{\theta 22}.\ \overline{(\!|(t, \rho, (u, \rho).\mathsf{C}'.\theta'')|\!)}\ () \Downarrow v'_{\theta 22} \Downarrow^{j_1'''} v_{\theta 22} \wedge (j - j') = (j_1'' - j_1''')$

$\wedge\ \forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22}$ (ET-2)

From (ET-0) and Definition 39, Definition 40 it suffices to prove that

$\exists j''', v'_{\theta 2}, v_{\theta 2}.\ \overline{(\!|((\!|t, \rho|\!)\ (\!|(u, \rho)|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))|\!)}\ () \Downarrow v'_{\theta 2} \Downarrow^{j'''} v_{\theta 2} \wedge (j - j') = (j'' - j''') \wedge$
$\forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$ (ET-3)

Since $(t, \rho, (u, \rho).\mathsf{C}'.\theta')$ is well typed therefore we know that

$\overline{(\!|((\!|t, \rho|\!)\ (\!|(u, \rho)|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))|\!)} =$

$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b' = e'_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$

where

$E_c = b'\ (coerce1\ !e'_{t2}\ c)\ d$

$e'_{t1} = \overline{(\!|((\!|t, \rho|\!)\ (\!|(u, \rho)|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!))|\!)}$

$e'_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$

From (ET-2) we know that $e'_{t1}() \Downarrow v'_{\theta 22} \Downarrow^{j_1'''} v_{\theta 22}$

and we need to prove that $v_{\theta 22}\ (coerce1\ !e'_{t2}\ c)\ d \Downarrow v_t \Downarrow^{j''' - j_1'''} v_{\theta 2}$ (ET-p)

Since we are given that $(\!|(t\ u, \rho, \mathsf{C}'.\theta')|\!)\ () \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$ this means from (ET-1.1) we have

$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
$\Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$

where

$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$

Also since we are given that $\overline{(\!|(t\ u, \rho, \mathsf{C}'.\theta'')|\!)}\ () \Downarrow v'_{\theta 11} \Downarrow^{j_1''} v_{\theta 11}$ this means we have

$e_{t1}\ () \Downarrow v'_{\theta 11} \Downarrow^{j_1''} v_{\theta 11}$

This means $v_{\theta 11}\ (coerce1\ !e_{t2}\ c)\ d \Downarrow - \Downarrow^{y} v_{\theta 1}$ for some y s.t $y + j_1'' = j''$

Since $\forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22}$ and $e_{t2} = e'_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ therefore from Definition 43 we get $\forall s. v_{\theta 1} \overset{s}{\approx}_{aV}$ $v_{\theta 2}$. Also from Definition 43 we have

$j'' - j_1'' = j''' - j_1''' =$

$j'' - j''' = j_1'' - j_1''' =$

$j'' - j''' = j - j'$ (From ET-IH)

$\square$

Lemma 52 (Cost and size lemma). $\forall e_s, D_s, E_s.$

$(e_s, \epsilon, \epsilon) \xrightarrow{*} D_s \rightarrow E_s\ \wedge$

D_s is well-typed $\wedge$

E_s is well-typed $\wedge$

$e_t = \overline{(\!|D_s|\!)} \wedge e_t\ () \Downarrow v_a \Downarrow^{j} v_1$

$\Longrightarrow$

$\exists e'_t.\ e'_t = \overline{(\!|E_s|\!)} \wedge e'_t\ () \Downarrow v_b \Downarrow^{j'} v_2 \wedge \forall s.\ v_1 \overset{s}{\approx}_{aE} v_2\ \wedge$

1. $j' = j \wedge |D_s| > |E_s|$ or
2. $j' = j - 1 \wedge |E_s| < |D_s| + |e_s|$

Proof. We case analyze on the $D_s \rightarrow E_s$ reduction

1. App1:
   Given $D_s = (t\ u, \rho, \theta)$ and $E_s = (t, \rho, (u, \rho).\theta)$
   Let $D'_s = (t\ u, \rho, \epsilon)$ and $E'_s = (t, \rho, (u, \rho).\epsilon)$
   Since we are given that D_s is well-typed and E_s is well-typed therefore from Lemma 53 we also have
   D'_s is well-typed and E'_s is well-typed

   Also since we know that $e_t\ () \Downarrow v_a \Downarrow^j v_1$ therefore from Lemma 54 we also know that
   $\exists j_e.\overline{(\!|D'_s|\!)}\ () \Downarrow v'_d \Downarrow^{j_e} v_d$
   From Lemma 50 we know that $\exists v_e.\ \overline{(\!|E'_s|\!)}\ () \Downarrow v'_e \Downarrow^{j_e} v_e$ s.t $\forall s.v_d \overset{s}{\approx}_{aV} v_e$

   And finally from Lemma 51 we know that $\overline{(\!|E_s|\!)}\ () \Downarrow v_b \Downarrow^j v_2$ s.t $\forall s.v_1 \overset{s}{\approx}_{aV} v_2$

   $|D_s| > |E_s|$ holds directly from the Definition of $| - |$
2. App2:
   Given: $(\lambda x.t, \rho, c.\theta) \rightarrow (t, c.\rho, \theta)$
   We induct on θ

   (a) Case $\theta = \epsilon$:
   Since we are given that D_s i.e $(\lambda x.t, \rho, c.\epsilon)$ is well typed
   Therefore from Theorem 42 $(\!|(\lambda x.t, \rho, c.\epsilon)|\!)$ is well-typed
   From Definition 40 $(\!|((\!|\lambda x.t, \rho|\!)\ (\!|c|\!), ., \epsilon)|\!)$ is well-typed
   Again from Definition 40 $(\!|\lambda x.t, \rho|\!)\ (\!|c|\!)$ is well-typed
   From Definition 39 we have
   $((\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_n|\!)\ (\!|c|\!))$ is well-typed

   Therefore from Theorem 22 we know that
   $\overline{(\!|D_s|\!)} =$
   $\overline{((\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_n|\!)\ (\!|\mathsf{C}|\!))} =$
   $\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
   where
   $E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
   $e_{t1} = \overline{((\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_n|\!))}$
   $e_{t2} = \overline{(\!|\mathsf{C}|\!)}$ (S-A0)

   Since we are given that $\overline{(\!|D_s|\!)}\ () \Downarrow v_a \Downarrow^j v_1$
   therefore from the evaluation rules we know that
   $\overline{(\!|t|\!)}[\overline{(\!|\mathsf{C}|\!)}()/x][\overline{(\!|\mathsf{C}_1|\!)}()/x_1] \dots [\overline{(\!|\mathsf{C}_1|\!)}()/x_1] \Downarrow - \Downarrow^j v_1$ (S-A0.1)

Similarly since we are given that E_s i.e $(t, c.\rho, \epsilon)$ is well-typed
Therefore from Theorem 42 $(\!|(t, c.\rho, \epsilon)|\!)$ is well-typed
From Definition 40 $(\!|t, c.\rho|\!)$ is well-typed
From Definition 39 we have $((\lambda x, x_1 \dots x_n.t)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_n|\!))$ is well-typed

Therefore from Theorem 22 we know that
$\overline{(\!|E_s|\!)} =$
$\overline{((\lambda x\ x_1 \dots x_n.t)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_n|\!))} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
$e_{t1} = \overline{((\lambda x\ x_1 \dots x_n.t)\ (\!|\mathsf{C}|\!), (\!|\mathsf{C}_1|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!))}$
$e_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ (S-A1)

From (SA-0.1) we know that
$\overline{(\!|E_s|\!)}\ () \Downarrow - \Downarrow^j v_1$
And finally from Theorem 47 we have $\forall s.v_1 \overset{s}{\approx}_{aV} v_1$

(b) Case $\theta = \mathsf{C}'.\theta'$:
Let $\theta' = \mathsf{C}_{\theta_1} \dots \mathsf{C}_{\theta_n}$ and $\rho = \mathsf{C}_{\rho_1} \dots \mathsf{C}_{\rho_n}$

Since we are given that D_s i.e $(\lambda x.t, \rho, \mathsf{C}.\mathsf{C}'.\theta')$ is well typed
Therefore from Theorem 42 we know that $(\!|(\lambda x.t, \rho, \mathsf{C}.\mathsf{C}'.\theta')|\!)$ is well-typed
From Definition 40 we also have $(\!|((\!|\lambda x.t, \rho|\!)\ (\!|\mathsf{C}|\!), ., \mathsf{C}'.\theta')|\!)$ is well-typed
which further means that $(\!|((\!|\lambda x.t, \rho|\!)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}'|\!), ., \theta')|\!)$ is well-typed
which further means that $(\!|(\!|\lambda x.t, \rho|\!)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}'|\!)\ (\!|\mathsf{C}_{\theta_1}|\!) \dots (\!|\mathsf{C}_{\theta_1}|\!)|\!)$ is well-typed
which further means that $(\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_{\rho_1}|\!) \dots (\!|\mathsf{C}_{\rho_n}|\!)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}'|\!)\ (\!|\mathsf{C}_{\theta_1}|\!) \dots (\!|\mathsf{C}_{\theta_m}|\!)$ is well-typed

From Theorem 22 we have
$\overline{(\!|D_s|\!)} = \overline{(\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_{\rho_1}|\!) \dots (\!|\mathsf{C}_{\rho_n}|\!)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}'|\!)\ (\!|\mathsf{C}_{\theta_1}|\!) \dots (\!|\mathsf{C}_{\theta_m}|\!)} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
$e_{t1} = \overline{(\lambda x_1 \dots x_n.\lambda x.t)\ (\!|\mathsf{C}_{\rho_1}|\!) \dots (\!|\mathsf{C}_{\rho_n}|\!)\ (\!|\mathsf{C}|\!)\ (\!|\mathsf{C}'|\!)\ (\!|\mathsf{C}_{\theta_1}|\!) \dots (\!|\mathsf{C}_{\theta_{m-1}}|\!)}$
$e_{t2} = \overline{(\!|\mathsf{C}_{\theta_m}|\!)}$ (S-A2)

Since we are given that $\overline{(\!|D_s|\!)}\ () \Downarrow v_a \Downarrow^j v_1$
therefore from the evaluation rules we know that
$\exists e', j_1.\overline{(\!|t|\!)}[\overline{(\!|\mathsf{C}|\!)}()/x][\overline{(\!|\mathsf{C}_1|\!)}()/x_1] \dots [\overline{(\!|\mathsf{C}_1|\!)}()/x_1] \Downarrow - \Downarrow^{j_1} \lambda x' x_1 \dots x_m.e'$
s.t
$\lambda x' x_1 \dots x_m.e' \overline{(\!|t|\!)}[\overline{(\!|\mathsf{C}'|\!)}()/x][\overline{(\!|\mathsf{C}_{\theta_1}|\!)}()/x_1] \dots [\overline{(\!|\mathsf{C}_{\theta_m}|\!)}()/x_m] \Downarrow - \Downarrow^{j_2} v_1$
and $j_1 + j_2 = j$ (S-A2.1)

Similarly since we are given that E_s i.e $(t, \mathsf{C}.\rho, \mathsf{C}'.\theta')$ is well typed
Therefore from Theorem 42 we know that $(\!|(t, \mathsf{C}.\rho, \mathsf{C}'.\theta')|\!)$ is well-typed

From Definition 40 we also have $(\!|((\!|t, \mathtt{C}.\rho|\!)\ (\!|\mathtt{C}'|\!), ., \theta')|\!)$ is well-typed
which further means that $(\!|((\!|t, \mathtt{C}.\rho|\!)\ (\!|\mathtt{C}'|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!))|\!)$ is well-typed
which further means that $(\lambda x, x_1 \ldots x_n.t)(\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}'|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$ is well-typed

From Theorem 22 we have
$\overline{(\!|E_s|\!)} = \overline{(\lambda x, x_1 \ldots x_n.t)(\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}'|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)} =$
$\lambda p.\mathsf{release} - = p \text{ in bind } a = \mathsf{store}() \text{ in bind } b = e_{t1}\ a \text{ in bind } c = \mathsf{store!}() \text{ in bind } d = \mathsf{store}() \text{ in } E_c$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
$e_{t1} = \overline{(\lambda x, x_1 \ldots x_n.t)(\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}'|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_{m-1}}|\!)}$
$e_{t2} = \overline{(\!|\mathtt{C}_{\theta_m}|\!)}$ (S-A3)

From (S-A2.1) it is clear that
$\overline{(\!|E_s|\!)}\ () \Downarrow - \Downarrow^j v_1$
And finally from Theorem 47 we have $\forall s.v_1 \overset{s}{\approx}_{aV} v_1$

$|D_s| > |E_s|$ holds directly from the Definition of $|-|$

3. Fix:

   Given: $(\mathsf{fix} x.t, \rho, \theta) \rightarrow (t, (\mathsf{fix} x.t, \rho).\rho, \theta)$
   Let $D'_s = (\mathsf{fix} x.t, \rho, \epsilon)$ and $E'_s = (t, (\mathsf{fix} x.t, \rho).\rho, \epsilon)$

   Since we are given that D_s and E_s are well-typed therefore from Lemma 53 we know that D'_s and E'_s are well-typed too.

   Also since we know that $e_t\ () \Downarrow v_a \Downarrow^j v_1$ therefore from Lemma 54 we also know that $\exists j_e.\overline{(\!|D'_s|\!)} \Downarrow - \Downarrow^{j_e} v_e$
   From Lemma 57 we know that $\overline{(\!|E'_s|\!)}\ () \Downarrow v'_e \Downarrow^{j_e} v_e$

   And then from Lemma 55 we know that $\overline{(\!|E_s|\!)} \Downarrow v_b \Downarrow^j v_2$ s.t $\forall s.v_1 \overset{s}{\approx}_{aV} v_2$

   $|D_s| > |E_s|$ holds directly from the Definition of $|-|$

4. Var:

   Given: $D_s = (x, (t_0, \rho_0) \ldots (t_x, \rho_x) \ldots (t_n, \rho_n), \theta)$ and $E_s = (t_x, \rho_x, \theta)$
   Let $D'_s = (x, (t_0, \rho_0) \ldots (t_x, \rho_x) \ldots (t_n, \rho_n), \epsilon)$ and $E'_s = (t_x, \rho_x, \epsilon)$

   Since we are given that D_s and E_s are well-typed therefore from Lemma 53 we know that D'_s and E'_s are well-typed too.

   Also since we know that $e_t\ () \Downarrow - \Downarrow^j v_1$ therefore from Lemma 54 we also know that $\exists j_e.\overline{(\!|D'_s|\!)} \Downarrow - \Downarrow^{j_e} v_e$
   From Lemma 59 we know that $\overline{(\!|E'_s|\!)} \Downarrow - \Downarrow^{j_e - 1} v_e$

And then from Lemma 58 we know that $\overline{\llparenthesis E_s \rrparenthesis} \Downarrow - \Downarrow^{j-1} v_2$ s.t $\forall s. v_1 \overset{s}{\approx}_{aV} v_2$

$|E_s| < |D_s| + |e_s|$ holds directly from the Definition of $|-|$ and from Lemma 4.2 in [3]

$\square$

Lemma 53 (ϵ typing)**.** $\forall \Theta, \Delta, I, e, \rho, \theta.$
$\Theta; \Delta \vdash_{-} (e, \rho, \theta) : - \implies \Theta; \Delta \vdash_{-} (e, \rho, \epsilon) : -$

Proof. Main derivation:

$$\dfrac{\dfrac{\dfrac{}{\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau}\ \text{Given}}{\Theta; \Delta \vdash_J (e, \rho) : \sigma}\ \text{By inversion} \qquad \dfrac{}{\Theta; \Delta \vdash_0 \epsilon : (\sigma, \sigma)}}{\Theta; \Delta \vdash_J (e, \rho, \epsilon) : \sigma}$$

$\square$

Lemma 54 (ϵ reduction)**.** $\forall e, \rho, \theta.$
(e, ρ, θ) is well typed $\wedge\ \overline{\llparenthesis (e, \rho, \theta) \rrparenthesis}\ () \Downarrow - \Downarrow^{-} - \implies \overline{\llparenthesis (e, \rho, \epsilon) \rrparenthesis}\ () \Downarrow - \Downarrow^{-} -$

Proof. Since (e, ρ, θ) is well typed therefore from Lemma 53 we also know that (e, ρ, ϵ) is well typed

From Theorem 42 we know that $\llparenthesis (e, \rho, \epsilon) \rrparenthesis$ is also well typed

From Definition 40 we know that $\llparenthesis (e, \rho, \epsilon) \rrparenthesis = \llparenthesis (e, \rho) \rrparenthesis$

Let $\theta = \mathtt{C}_1 \ldots \mathtt{C}_n$
Similarly from Definition 40 we also know that
$\llparenthesis (e, \rho, \theta) \rrparenthesis = \llparenthesis (e, \rho, \mathtt{C}_1 \ldots \mathtt{C}_n) \rrparenthesis =$
$\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis, [], \mathtt{C}_2 \ldots \mathtt{C}_n) \rrparenthesis =$
$\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis \ldots \llparenthesis \mathtt{C}_n \rrparenthesis, [], \epsilon) \rrparenthesis =$
$\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis \ldots \llparenthesis \mathtt{C}_n \rrparenthesis) \rrparenthesis$

From Theorem 22 we know that
$\overline{\llparenthesis (e, \rho, \theta) \rrparenthesis} =$
$\overline{\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis \ldots \llparenthesis \mathtt{C}_n \rrparenthesis) \rrparenthesis} =$
$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\ a = \mathsf{store}() \text{ in } \mathsf{bind}\ b = e_{t1}\ a \text{ in } \mathsf{bind}\ c = \mathsf{store!}() \text{ in } \mathsf{bind}\ d = \mathsf{store}() \text{ in } E_c$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
$e_{t1} = \overline{\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis \ldots \llparenthesis \mathtt{C}_{n-1} \rrparenthesis) \rrparenthesis}$
$e_{t2} = \overline{\llparenthesis \mathtt{C}_n \rrparenthesis}$ (E0)

Since $\overline{\llparenthesis (\llparenthesis e, \rho \rrparenthesis\ \llparenthesis \mathtt{C}_1 \rrparenthesis \ldots \llparenthesis \mathtt{C}_n \rrparenthesis) \rrparenthesis} \Downarrow - \Downarrow^{-} -$, therefore we also know that $\overline{\llparenthesis \llparenthesis e, \rho \rrparenthesis \rrparenthesis} \Downarrow - \Downarrow^{-} -$

$\square$

Lemma 55 (Lemma for fix : non-empty stack)**.** $\forall t, \rho, \theta, j, j', j'', v_{\epsilon 1}, v_{\epsilon 2}, v_{\theta 1}.$
$(\mathsf{fix} x.t, \rho, \epsilon)$ and $(t, (\mathsf{fix} x.t, \rho).\rho, \epsilon)$ are well-typed
$(\mathsf{fix} x.t, \rho, \theta)$ and $(t, (\mathsf{fix} x.t, \rho).\rho, \theta)$ are well-typed
$\overline{\llparenthesis (\mathsf{fix} x.t, \rho, \epsilon) \rrparenthesis}\ () \Downarrow - \Downarrow^{j} v_{\epsilon 1} \wedge \overline{\llparenthesis (t, (\mathsf{fix} x.t, \rho).\rho, \epsilon) \rrparenthesis}\ () \Downarrow - \Downarrow^{j'} v_{\epsilon 1} \wedge \forall s. v_{\epsilon 1} \overset{s}{\approx}_{aV} v_{\epsilon 2} \wedge$
$\overline{\llparenthesis (\mathsf{fix} x.t, \rho, \theta) \rrparenthesis}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1} \wedge$
$\implies$
$\exists v_{\theta 2}, j'''.\ \overline{\llparenthesis (t, (\mathsf{fix} x.t, \rho).\rho, \theta) \rrparenthesis}\ () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''')$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:

   Directly from given

2. Case $\theta = \mathsf{C}'.\theta'$:

   Let $\theta' = \mathsf{C}'_1 \dots \mathsf{C}'_n$ and $\theta'' = \mathsf{C}'_1 \dots \mathsf{C}'_{n-1}$

   Given:
   $(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')$ and $(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta')$ are well-typed $\wedge$
   $\overline{(\!|(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1}$
   We need to prove that
   $\overline{(\!|(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''')$ (ET-0)

   From IH we know
   $(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta'')$ and $(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta'')$ are well-typed,
   $\overline{(\!|(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta'')|\!)}\ () \Downarrow - \Downarrow^{j''_1} v_{\theta 11} \implies$
   $\overline{(\!|(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta'')|\!)}\ () \Downarrow - \Downarrow^{j'''_1} v_{\theta 22} \wedge \forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j''_1 - j'''_1)$
   (ET-IH)

   From Definition 39 and Definition 40 we know that
   $(\!|(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')|\!) = (\!|\mathsf{fix}x.t, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!)$ (ET-1)

   Since $(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')$ is well typed therefore we know that
   $\overline{(\!|(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')|\!)} = \overline{(\!|\mathsf{fix}x.t, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!)} =$
   $\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
   where
   $E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
   $e_{t1} = \overline{(\!|\mathsf{fix}x.t, \rho|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)}$
   $e_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ (ET-1.1)

   Since we know that $\overline{(\!|(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1}$ therefore we also know that
   $\exists j''_1, v'_1.e_{t1}() \Downarrow - \Downarrow^{j''_1} v_{\theta 11}$

   Also since we know that
   $(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta')$ and $(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta')$ are well-typed
   therefore from Lemma 56 we also know that
   $(\mathsf{fix}x.t, \rho, \mathsf{C}'.\theta'')$ and $(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta'')$ are well-typed

   Therefore from (ET-IH) we have
   $\exists v_{\theta 22}, j'''_1.\ \overline{(\!|(t, (\mathsf{fix}x.t, \rho).\rho, \mathsf{C}'.\theta'')|\!)} \Downarrow - \Downarrow^{j'''_1} v_{\theta 22} \wedge \forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j''_1 - j'''_1)$
   (ET-2)

From Definition 39 we know that
$\overline{(\!|(t, (\mathsf{fix} x.t, \rho).\rho, \mathsf{C}'.\theta')|\!)} = \overline{((\!|t, (\mathsf{fix} x.t, \rho).\rho|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))}$

Since $(t, (\mathsf{fix} x.t, \rho).\rho, \mathsf{C}'.\theta')$ is well typed therefore we know that
$\overline{(\!|(t, (\mathsf{fix} x.t, \rho).\rho, \mathsf{C}'.\theta')|\!)} =$
$\overline{((\!|t, (\mathsf{fix} x.t, \rho).\rho|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!))} =$
$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b' = e'_{t1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store}!() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c$
where
$E_c = b'\ (coerce1\ !e'_{t2}\ c)\ d$
$e'_{t1} = \overline{((\!|t, (\mathsf{fix} x.t, \rho).\rho|\!)\ (\!|\mathsf{C}'|\!) \ldots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_{n-1}|\!))}$
$e'_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$

Since from (ET-2) we know that $\overline{(\!|(t, (\mathsf{fix} x.t, \rho).\rho, \mathsf{C}'.\theta'')|\!)} \Downarrow - \Downarrow^{j'''_1} v_{\theta 22}$

Therefore it suffices to prove that
$v_{\theta 22}\ (coerce1\ !e'_{t2}\ c)\ d \Downarrow - \Downarrow^{j'''-j'''_1} v_{\theta 2}$ and $\forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$ (ET-p)

Since we are given that $\overline{(\!|(\mathsf{fix} x.t, \rho, \mathsf{C}'.\theta')|\!)}$ this means from (ET-1.1) we have
$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store}!() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c \Downarrow - \Downarrow^{j''} v_{\theta 1}$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$

This means
1) $e_{t1}\ () \Downarrow - \Downarrow^{j''_1} v_{\theta 11}$ and
2) This means $v_{\theta 11}\ (coerce1\ !e_{t2}\ c)\ d \Downarrow - \Downarrow^{y} v_{\theta 1}$ for some y s.t $y + j''_1 = j''$

Since from (ET-2) we know that $\forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22}$ and since $e_{t2} = e'_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ therefore from Definition 43 and Lemma 49 we have
$v_{\theta 22}\ (coerce1\ !e'_{t2}\ c)\ d \Downarrow - \Downarrow^{j''-j''_1} v_{\theta 2}$ and $\forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$
This means
$j'' - j''_1 = j''' - j'''_1 =$
$j'' - j''' = j''_1 - j'''_1 =$
$j'' - j''' = j - j'$ (From IH)

∎

Lemma 56. $\forall \mathsf{C}, \theta.$
$\theta.\mathsf{C}$ is well-typed $\implies$ θ is well-typed

Proof. Proof by induction on θ

1. Base case $\theta = \epsilon$:

   Directly from the typing rule for ϵ

2. Case $\theta = \mathtt{C}'.\theta'$

   This means we have $\mathtt{C}'.\theta'.\mathtt{C}$ is well-typed. This means from the stack typing rule for closure we know that $\theta'.\mathtt{C}$ is well-typed.

   From IH we know that θ' is well-typed.

   Since $\mathtt{C}'$ is well tped and θ' is well-typed therefore $\mathtt{C}'.\theta'$ is well-typed.

$\square$

Lemma 57 (Lemma for fix : empty stack)**.** $\forall t, \rho, \theta.$
$\llparenthesis(\mathsf{fix}x.t, \rho, \epsilon)\rrparenthesis$ is well-typed $\wedge$
$\llparenthesis(t, (\mathsf{fix}x.t, \rho).\rho, \epsilon)\rrparenthesis$ is well-typed $\wedge$
$\overline{\llparenthesis(\mathsf{fix}x.t, \rho, \epsilon)\rrparenthesis}\ () \Downarrow - \Downarrow^j v_1 \implies$
$\overline{\llparenthesis(t, (\mathsf{fix}x.t, \rho).\rho, \epsilon)\rrparenthesis}() \Downarrow - \Downarrow^j v_2 \wedge \forall s. v_1 \overset{s}{\approx}_{aV} v_2$

Proof. Let $\rho = (\mathtt{C}_1, \ldots, \mathtt{C}_n)$
Since we know that $\llparenthesis(\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis$ is well-typed and
$\llparenthesis(\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis = ((\lambda x_1 \ldots x_n.\mathsf{fix}x.t)\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)$
Therefore from Theorem 22 we know that
$\overline{\llparenthesis(\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis} =$
$\overline{((\lambda x_1 \ldots x_n.\mathsf{fix}x.t)\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2}\ c)\ d$
$e_{t1} = \overline{((\lambda x_1 \ldots x_n.\mathsf{fix}x.t)\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_{n-1}\rrparenthesis)}$
$e_{t2} = \overline{\llparenthesis\mathtt{C}_n\rrparenthesis}$ (F1)

Since we know that
$\overline{((\lambda x_1 \ldots x_n.\mathsf{fix}x.t)\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)}\ () \Downarrow - \Downarrow^j v_1$
Therefore from E-release, E-store, E-bind, E-subExpE and E-app we know that
$\bar{t}[\mathsf{fix}x.\bar{t}[\overline{\llparenthesis\mathtt{C}_1\rrparenthesis}()/x_1] \ldots [\overline{\llparenthesis\mathtt{C}_n\rrparenthesis}()/x_n]\ ()/x] \Downarrow - \Downarrow^j v_1$ (F2)

Similarly since we know that $\llparenthesis(t, (\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)).(\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis$ is well-typed and
$\llparenthesis(t, (\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)).(\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis = ((\lambda x, x_1 \ldots x_n.t)\ \llparenthesis\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)\rrparenthesis\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)$
Therefore from Theorem 22 we know that
$\overline{\llparenthesis(t, (\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)).(\mathtt{C}_1, \ldots, \mathtt{C}_n), \epsilon)\rrparenthesis} =$
$\overline{((\lambda x, x_1 \ldots x_n.t)\ \llparenthesis\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)\rrparenthesis\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e'_{t1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e'_{t2}\ c)\ d$
$e'_{t1} = \overline{((\lambda x, x_1 \ldots x_n.t)\ \llparenthesis\mathsf{fix}x.t, (\mathtt{C}_1, \ldots, \mathtt{C}_n)\rrparenthesis\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_{n-1}\rrparenthesis)}$
$e'_{t2} = \overline{\llparenthesis\mathtt{C}_n\rrparenthesis}$ (F3)

We need to prove that
$\overline{((\lambda x, x_1 \ldots x_n.t)\ \llparenthesis\mathsf{fix}x.t, \rho\rrparenthesis\ \llparenthesis\mathtt{C}_1\rrparenthesis \ldots\ \llparenthesis\mathtt{C}_n\rrparenthesis)} \Downarrow - \Downarrow^j v_2$

This means it suffices to prove that
$\bar{t}[\mathsf{fix}x.\bar{t}[\overline{\llparenthesis\mathtt{C}_1\rrparenthesis}()/x_1] \ldots [\overline{\llparenthesis\mathtt{C}_n\rrparenthesis}()/x_n]\ ()/x] \Downarrow - \Downarrow^j v_2$
We get this directly from (F2) and Lemma 48 $\square$

Lemma 58 (Lemma for var : non-empty stack)**.** $\forall t, \rho, \theta, j, j', j'', v_{\epsilon 1}, v_{\epsilon 2}, v_{\theta 1}$.

$(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon)$ and (t_x, ρ_x, ϵ) are well-typed

$\underline{(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \theta)}$ and $\underline{(t, (\mathsf{fix} x.t, \rho).\rho, \theta)}$ are well-typed

$\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon)|\!)}\ () \Downarrow - \Downarrow^{j} v_{\epsilon 1} \wedge \overline{(\!|(t_x, \rho_x, \epsilon)|\!)}\ () \Downarrow - \Downarrow^{j'} v_{\epsilon 1} \wedge$

$\forall s. v_{\epsilon 1} \overset{s}{\approx}_{aV} v_{\epsilon 2} \wedge$

$\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \theta)|\!)}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1} \wedge$

$\Longrightarrow$

$\exists v_{\theta 2}, j'''.\ \overline{(\!|(t_x, \rho_x, \theta)|\!)}\ () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''')$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:

   Directly from given

2. Case $\theta = \mathsf{C}'.\theta'$:

   Let $\theta' = \mathsf{C}'_1 \dots \mathsf{C}'_n$ and $\theta'' = \mathsf{C}'_1 \dots \mathsf{C}'_{n-1}$

   Given:

   $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')$ and $(t_x, \rho_x, \mathsf{C}'.\theta')$ are well-typed $\wedge$

   $\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1}$

   We need to prove that

   $\overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''')$ (ET-0)

   From IH we know

   $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta'')$ and $(t_x, \rho_x, \mathsf{C}'.\theta'')$ are well-typed,

   $\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta'')|\!)}\ () \Downarrow - \Downarrow^{j''_1} v_{\theta 11} \Longrightarrow$

   $\overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta'')|\!)}\ () \Downarrow - \Downarrow^{j'''_1} v_{\theta 22} \wedge \forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j''_1 - j'''_1)$ (ET-IH)

   From Definition 39 and Definition 40 we know that

   $(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')|\!) = (\!|x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n)|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!)$
   (ET-1)

   Since $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')$ is well typed therefore we know that

   $\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')|\!)} = \overline{(\!|x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n)|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)\ (\!|\mathsf{C}_n|\!)}$
   $=$

   $\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t1}\ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c$

   where

   $E_c = b\ (coerce1\ !e_{t2}\ c)\ d$

   $e_{t1} = \overline{(\!|x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n)|\!)\ (\!|\mathsf{C}'|\!) \dots\ (\!|\mathsf{C}_{n-1}|\!)}$

   $e_{t2} = \overline{(\!|\mathsf{C}_n|\!)}$ (ET-1.1)

   Since we know that $\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')|\!)}\ () \Downarrow - \Downarrow^{j''} v_{\theta 1}$ therefore we also know that

$\exists j_1'', v_1'.e_{t1}() \Downarrow - \Downarrow^{j_1''} v_1'$

Also since we know that

$(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')$ and $(t, (\mathsf{fix} x.t, \rho).\rho, \mathsf{C}'.\theta')$ are well-typed

therefore from Lemma 56 we also know that

$(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta'')$ and $(t_x, \rho_x, \mathsf{C}'.\theta'')$ are well-typed

Therefore from (ET-IH) we have

$\exists v_{\theta 22}, j_1'''. \ \overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta'')|\!)} \Downarrow - \Downarrow^{j_1'''} v_{\theta 22} \wedge \forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j_1'' - j_1''')$ (ET-2)

From Definition 39 we know that

$\overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta')|\!)} = \overline{((\!|t_x, \rho_x|\!) \ (\!|\mathsf{C}'|\!) \dots \ (\!|\mathsf{C}_{n-1}|\!) \ (\!|\mathsf{C}_n|\!))}$

Since $(t_x, \rho_x, \mathsf{C}'.\theta')$ is well typed therefore we know that

$\overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta')|\!)} =$

$\overline{((\!|t_x, \rho_x|\!) \ (\!|\mathsf{C}'|\!) \dots \ (\!|\mathsf{C}_{n-1}|\!) \ (\!|\mathsf{C}_n|\!))} =$

$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b' = e_{t1}' \ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c$

where

$E_c = b' \ (coerce1 \ !e_{t2}' \ c) \ d$

$e_{t1}' = \overline{((\!|t_x, \rho_x|\!) \ (\!|\mathsf{C}'|\!) \dots \ (\!|\mathsf{C}_{n-1}|\!) \ (\!|\mathsf{C}_{n-1}|\!))}$

$e_{t2}' = \overline{(\!|\mathsf{C}_n|\!)}$

Since from (ET-2) we know that $\overline{(\!|(t_x, \rho_x, \mathsf{C}'.\theta'')|\!)} \Downarrow - \Downarrow^{j_1'''} v_{\theta 22}$

Therefore it suffices to prove that

$v_{\theta 22} \ (coerce1 \ !e_{t2}' \ c) \ d \Downarrow - \Downarrow^{j''' - j_1'''} v_{\theta 2}$ and $\forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$ (ET-p)

Since we are given that $(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta')|\!) \Downarrow - \Downarrow^{j''} v_{\theta 1}$ this means from (ET-1.1) we have

$\lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = e_{t1} \ a \text{ in } \mathsf{bind}\, c = \mathsf{store!}() \text{ in } \mathsf{bind}\, d = \mathsf{store}() \text{ in } E_c \Downarrow - \Downarrow^{j''} v_{\theta 1}$

where

$E_c = b \ (coerce1 \ !e_{t2} \ c) \ d$

This means

1) $\overline{(\!|(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathsf{C}'.\theta'')|\!)} \ () \Downarrow - \Downarrow^{j_1''} v_{\theta 11}$ and

2) This means $v_{\theta 11} \ (coerce1 \ !e_{t2} \ c) \ d \Downarrow - \Downarrow^{y} v_{\theta 1}$ for some y s.t $y + j_1'' = j''$

Since from (ET-2) we have $\forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \wedge$ and since $e_{t2} = e_{t2}' = \overline{(\!|\mathsf{C}_n|\!)}$ therefore from Definition 43 and Lemma 49 we have

$v_{\theta 22}\ (coerce1\ !e'_{t2}\ c)\ d \Downarrow - \Downarrow^{j''-j''_1} v_{\theta 2}$ and $\forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$

This means

$j'' - j''_1 = j''' - j'''_1 =$

$j'' - j''' = j''_1 - j'''_1 =$

$j'' - j''' = j - j'$ (From IH)

$\square$

Lemma 59 (Lemma for var : empty stack)**.** $\forall t, \rho, \theta.$

$\Theta; \Delta; . \vdash_{-} \llparenthesis(x, (t_0, \rho_0) \ldots (t_x, \rho_x) \ldots (t_n, \rho_n), \epsilon)\rrparenthesis : - \ \wedge$
$\Theta; \Delta; . \vdash_{-} \llparenthesis(t_x, \rho_x, \epsilon)\rrparenthesis : - \ \wedge$
$\overline{\llparenthesis(x, (t_0, \rho_0) \ldots (t_x, \rho_x) \ldots (t_n, \rho_n), \epsilon)\rrparenthesis}\ () \Downarrow - \Downarrow^{j} v \implies$
$\overline{\llparenthesis(t_x, \rho_x, \epsilon)\rrparenthesis}\ () \Downarrow - \Downarrow^{j-1} v$

Proof. From Definition 40 we also have

$\llparenthesis(x, (t_0, \rho_0), \ldots (t_x, \rho_x), \ldots (t_n, \rho_n), \epsilon)\rrparenthesis$
$= \llparenthesis x, (t_0, \rho_0), \ldots (t_x, \rho_x), \ldots (t_n, \rho_n)\rrparenthesis$
$= (\lambda x_1 \ldots x \ldots x_n.x)\ \llparenthesis(t_0, \rho_0)\rrparenthesis \ldots\ \llparenthesis(t_n, \rho_n)\rrparenthesis$

Similarly from Definition 40 we also have
$\llparenthesis(t_x, \rho_x, \epsilon)\rrparenthesis = \llparenthesis(t_x, \rho_x)\rrparenthesis$ (S-V1)

Therefore from Theorem 22 we know that
$\overline{\llparenthesis(x, ((t_1, \rho_1), \ldots (t_x, \rho_x) \ldots, (t_n, \rho_n)), \epsilon)\rrparenthesis} =$
$\overline{((\lambda x_1 \ldots x \ldots x_n.x)\ \llparenthesis(t_1, \rho_1)\rrparenthesis \ldots \llparenthesis(t_x, \rho_x)\rrparenthesis \ldots\ \llparenthesis(t_n, \rho_n)\rrparenthesis)} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,n}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ \overline{(coerce1\ !e_{t2,n}\ c)\ d}$
$e_{t1,n} = \overline{((\lambda x_1 \ldots x \ldots x_n.x)\ \llparenthesis(t_1, \rho_1)\rrparenthesis \ldots \llparenthesis(t_x, \rho_x)\rrparenthesis \ldots\ \llparenthesis(t_{n-1}, \rho_{n-1})\rrparenthesis)}$
$e_{t2,n} = \overline{\llparenthesis(t_n, \rho_n)\rrparenthesis}$ (V4)

Simialrly
$e_{t1,n} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,n-1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ (coerce1\ !e_{t2,n-1}\ c)\ d$
$e_{t1,n-1} = \overline{((\lambda x_1 \ldots x \ldots x_n.x)\ \llparenthesis(t_1, \rho_1)\rrparenthesis \ldots \llparenthesis(t_x, \rho_x)\rrparenthesis \ldots\ \llparenthesis(t_{n-2}, \rho_{n-2})\rrparenthesis)}$
$e_{t2,n-1} = \overline{\llparenthesis(t_{n-1}, \rho_{n-1})\rrparenthesis}$

In the same way we have
$e_{t1,1} =$
$\lambda p.\mathsf{release} - = p \mathsf{\ in\ bind\ } a = \mathsf{store}() \mathsf{\ in\ bind\ } b = e_{t1,1}\ a \mathsf{\ in\ bind\ } c = \mathsf{store!}() \mathsf{\ in\ bind\ } d = \mathsf{store}() \mathsf{\ in\ } E_c$
where
$E_c = b\ \overline{(coerce1\ !e_{t2,1}\ c)\ d}$
$e_{t1,1} = \overline{(\lambda x_1 \ldots x \ldots x_n.x)}$
$e_{t2,1} = \overline{\llparenthesis(t_1, \rho_1)\rrparenthesis}$

Similalry we also get
$e_{t1,1} =$

$\lambda p_1.\, \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\, !\, x = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{l,1}\ a$
where
$e_{l,1} = \overline{((\lambda x_2 \ldots x \ldots x_n.x))}$

and
$e_{l,n} =$
$\lambda p_1.\, \mathsf{ret}\, \lambda y.\lambda p_2.\, \mathsf{let}\, !\, x = y \text{ in } \mathsf{release} - = p_1 \text{ in } \mathsf{release} - = p_2 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } e_{l,n}\ a$
where
$e_{l,n} = \overline{x} = \lambda p.\mathsf{release} - = p \text{ in } \mathsf{bind} - = \uparrow^1 \text{ in } x$

Since we know that
$\overline{((\lambda x_1 \ldots x \ldots x_n.x)\ (\!|(t_0, \rho_0)|\!) \ldots\ (\!|(t_n, \rho_n)|\!))}\ ()\Downarrow - \Downarrow^j v$
this means from E-release, E-bind, E-store, E-app that
$(\mathsf{bind} - = \uparrow^1 \text{ in } \overline{(\!|(t_x, \rho_x)|\!)}\ ()) \Downarrow - \Downarrow^j v$
Therefore from E-bind, E-step and E-app we know that $\overline{(\!|(t_x, \rho_x)|\!)}\ () \Downarrow - \Downarrow^{j-1} v$ □

Theorem 60 (Rederiving dlPCF's soundness)**.** $\forall t, I, \tau, \rho.$
$\vdash_I (t, \epsilon, \epsilon) : \tau \wedge (t, \epsilon, \epsilon) \xrightarrow{n} (v, \rho, \epsilon) \implies n \leqslant |t| * (I + 1)$

Proof. Let us rename t to t_1 and v to t_{n+1} then we know that
$(t_1, \epsilon, \epsilon) \to (t_2, \rho_2, \theta_2) \ldots (t_n, \rho_n, \theta_n) \to (t_{n+1}, \rho, \epsilon)$

Since we are given that (t, ϵ, ϵ) is well-typed therefore from dlPCF's subject reduction we know that (t_2, ρ_2, θ_2) to (t_n, ρ_n, θ_n) and $(t_{n+1}, \rho, \epsilon)$ are all well-typed.

From Theorem 63 we know that $\forall 1 \leqslant i \leqslant n.(\!|(t_i, \rho_i, \theta_i)|\!) \xrightarrow{*} -$
Also from Theorem 42 we know that $\forall 1 \leqslant i \leqslant n.(\!|(t_i, \rho_i, \theta_i)|\!)$ is well typed

So now we can apply Theorem 36 and from Definition 34 to get
$\forall 1 \leqslant i \leqslant n+1.\exists j_i.\overline{(\!|(t_i, \rho_i, \theta_i)|\!)}\ () \Downarrow - \Downarrow^{j_i} -$

Next we apply Theorem 52 for every step of the reduction starting from $(t_1, \epsilon, \epsilon)$ and we know that either the cost reduces by 1 and the size increases by $|t|$ or cost remains the same and the size reduces.

Thus we know that size can vary from t to 1 and cost can vary from j_1 to 0. Therefore, the number of reduction steps are bounded by $|t| * (j_1 + 1)$
From Theorem 20 we know that $j_1 < I$ therefore we have $n \leqslant |t| * (I + 1)$ □

1.5.4 Cross-language model: Krivine to dlPCF

Definition 61 (Cross language logical realtion: Krivine to dlPCF)**.**

$$(v_k, \rho, \epsilon) \sim_v v_d \triangleq v_d = v_k\ \rho$$

$$(e_k, \rho, \theta) \sim_e e_d \triangleq \forall v_k, \rho'.(e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon) \implies \exists v_d.e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$$

Lemma 62. $\forall e_k, \rho, \theta, e'_k, \rho', \theta'.$
$(e_k, \rho, \theta) \xrightarrow{*} (e'_k, \rho', \theta') \implies \exists e'_d.(\!|(e_k, \rho, \theta)|\!) \xrightarrow{*} e'_d \wedge e'_d = (\!|(e'_k, \rho', \theta')|\!)$

Proof. Given: $(e_k, \rho, \theta) \xrightarrow{*} (e'_k, \rho', \theta')$

To prove: $\exists e'_d . (\!|(e_k, \rho, \theta)|\!) \xrightarrow{*} e'_d \wedge e'_d = (\!|(e'_k, \rho', \theta')|\!)$

Lets assume it takes n steps for $(e_k, \rho, \theta) \xrightarrow{n} (e'_k, \rho', \theta')$

We induct on n

<u>Base case ($n = 1$)</u>

1. App1:

   In this case we are given $(t\ u, \rho, \theta) \rightarrow (t, \rho, (u, \rho).\theta)$

   Let $\rho = \mathtt{C}_{\rho_1} \ldots \mathtt{C}_{\rho_n}$ and $\theta = \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   From Definition 40 we know that

   $(\!|(e_k, \rho, \theta)|\!) =$

   $(\lambda x_1 \ldots x_n . t\ u)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$

   From dlPCF's app rule we know that

   $(\lambda x_1 \ldots x_n . t\ u)\mathtt{C}_{\rho_1} \ldots \mathtt{C}_{\rho_n}\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m} \xrightarrow{*}$

   $t[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n]\ u[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n]\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$

   We choose e'_d as $t[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n]\ u[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n]\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$ and we get thedesired from Definition 40

2. App2:

   In this case we are given $(\lambda x.t, \rho, \mathtt{C}.\theta) \rightarrow (t, \mathtt{C}.\rho, \theta)$

   Let $\rho = \mathtt{C}_{\rho_1} \ldots \mathtt{C}_{\rho_n}$ and $\theta = \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   From Definition 40 we know that

   $(\!|(\lambda x.t, \rho, \mathtt{C}.\theta)|\!) =$

   $(\lambda x_1 \ldots x_n . \lambda x.t)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$

   From dlPCF's app rule we know that

   $(\lambda x_1 \ldots x_n . \lambda x.t)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!) \xrightarrow{*}$

   $t[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n][(\!|\mathtt{C}|\!)/x]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   We choose e'_d as $t[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n][(\!|\mathtt{C}|\!)/x]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$ and we get the desired from Definition 40

3. Var:

   In this case we are given $(x, (t_0, \rho_0) \ldots (t_n, \rho_n), \theta) \rightarrow (t_x, \rho_x, \theta)$

   Let $\theta = \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   From Definition 40 we know that

   $(\!|(x, (t_0, \rho_0) \ldots (t_n, \rho_n), \theta)|\!) =$

   $(\lambda x_1 \ldots x_n . \lambda x.t)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$

   From dlPCF's app rule we know that

$(\lambda x_1 \ldots x_n.x)(\!|(t_0, \rho_0)|\!) \ldots (\!|(t_n, \rho_n)|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!) \xrightarrow{*}$
$(\!|(t_x, \rho_x)|\!)\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

Let $\rho_x = \mathtt{C}_{x_1} \ldots \mathtt{C}_{x_k}$ therefore from Definition 40 we know that
$(\!|(t_x, \rho_x)|\!)\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m} =$
$\lambda x_{x_1} \ldots x_{x_k}.t_x\ (\!|\mathtt{C}_{x_1}|\!) \ldots (\!|\mathtt{C}_{x_k}|\!)\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

Therefore from dlPCF's app rule we know that
$(\!|(t_x, \rho_x)|\!)\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m} \xrightarrow{*} t_x[(\!|\mathtt{C}_{x_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{x_k}|\!)/x_k]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

We choose e'_d as $t_x[(\!|\mathtt{C}_{x_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{x_k}|\!)/x_k]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$ and we get the desired from Definition 40

4. Fix:

   In this case we are given $(\mathsf{fix}x.t, \rho, \theta) \rightarrow (t, (\mathsf{fix}x.t, \rho).\rho, \theta)$
   Let $\rho = \mathtt{C}_{\rho_1} \ldots \mathtt{C}_{\rho_n}$ and $\theta = \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   From Definition 40 we know that
   $(\!|(\mathsf{fix}x.t, \rho, \theta)|\!) =$
   $(\lambda x_1 \ldots x_n.\mathsf{fix}x.t)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|(\mathsf{fix}x.t, \rho)|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!)$

   From dlPCF's app and fix rule we know that
   $(\lambda x_1 \ldots x_n.\mathsf{fix}x.t)(\!|\mathtt{C}_{\rho_1}|\!) \ldots (\!|\mathtt{C}_{\rho_n}|\!)\ (\!|\mathtt{C}|\!)\ (\!|\mathtt{C}_{\theta_1}|\!) \ldots (\!|\mathtt{C}_{\theta_m}|\!) \xrightarrow{*}$
   $\mathsf{fix}x.t[(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n][(\!|(\mathsf{fix}x.t, \rho)|\!)/x]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m} \rightarrow$
   $t\ [(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n][(\!|(\mathsf{fix}x.t, \rho)|\!)/x]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$

   We choose e'_d as $t\ [(\!|\mathtt{C}_{\rho_1}|\!)/x_1] \ldots [(\!|\mathtt{C}_{\rho_n}|\!)/x_n][(\!|(\mathsf{fix}x.t, \rho)|\!)/x]\ \mathtt{C}_{\theta_1} \ldots \mathtt{C}_{\theta_m}$ and we get the desired from Definition 40

   <u>Inductive case</u>

   We get this directly from IH and the base case

□

Theorem 63 (Fundamental theorem). $\forall e_k, \rho, \theta.\ (e_k, \rho, \theta) \sim_e (\!|(e_k, \rho, \theta)|\!)$

Proof. From Definition 61 it suffices to prove that
$\forall v_k, \rho'.(e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon) \implies \exists v_d.e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$

This means athat given some v_k, ρ' s.t $(e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon)$ it suffices to prove that
$\exists v_d.e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$

From Lemma 62 we know that
$\exists e'_d.(\!|(e_k, \rho, \theta)|\!) \xrightarrow{*} e'_d \wedge e'_d = (\!|(v_k, \rho', \epsilon)|\!)$

Let $\rho' = \mathtt{C}_1 \ldots \mathtt{C}_n$ therefore from Definition 40 we know that
$(\!|(v_k, \rho', \epsilon)|\!) = (\lambda x_1 \ldots x_n v_k)\ (\!|\mathtt{C}_1|\!) \ldots (\!|\mathtt{C}_n|\!)$

Therefore from dlPCF's app rule we know that
$(\!|(v_k, \rho', \epsilon)|\!) \xrightarrow{*} v_k[(\!|\mathtt{C}_1|\!)/x_1] \ldots [(\!|\mathtt{C}_n|\!)/x_n]$

We choose v_d as $v_k[(\!|\mathtt{C}_1|\!)/x_1] \ldots [(\!|\mathtt{C}_n|\!)/x_n]$ and we get the desired from Definition 61

□

2 Development for univariate RAML's embedding

2.1 Syntax

Expressions	e	$::=$	$v \mid e_1\ e_2 \mid \langle\!\langle e_1, e_2 \rangle\!\rangle \mid \mathsf{let}\langle\!\langle x, y \rangle\!\rangle = e_1 \mathsf{\ in\ } e_2 \mid$
			$\langle e, e \rangle \mid \mathsf{fst}(e) \mid \mathsf{snd}(e) \mid \mathsf{inl}(e) \mid \mathsf{inr}(e) \mid \mathsf{case}\ e \mathsf{\ of\ } e; e \mid$
			$\mathsf{let}\,!\,x = e_1 \mathsf{\ in\ } e_2 \mid e :: e \mid e\ [] \mid e; x.e$
Values	v	$::=$	$x \mid () \mid c \mid \lambda x.e \mid \langle\!\langle v_1, v_2 \rangle\!\rangle \mid \langle v, v \rangle \mid \mathsf{inl}(e) \mid \mathsf{inr}(e) \mid\ !\,e \mid \mathit{nil} \mid$
			$\Lambda.e \mid \mathsf{ret}\, e \mid \mathsf{bind}\, x = e_1 \mathsf{\ in\ } e_2 \mid \uparrow^I \mid \mathsf{release}\, x = e_1 \mathsf{\ in\ } e_2 \mid \mathsf{store}\, e$
			(No value forms for $[I]\,\tau$)
Index	I	$::=$	$i \mid N \mid R \mid I + I \mid I - I \mid \sum_{i<I} I \mid \lambda_s i : S . I \mid I\ I$
Sort	S	$::=$	$\mathbb{N} \mid \mathbb{R}^+ \mid S \to S$
Kind	K	$::=$	$\mathit{Type} \mid S \to K$
Types	τ	$::=$	$\mathbf{1} \mid \mathsf{b} \mid \tau_1 \multimap \tau_2 \mid \tau_1 \otimes \tau_2 \mid \tau_1 \mathbin{\&} \tau_2 \mid \tau_1 \oplus \tau_2 \mid\ !\,\tau \mid [I]\,\tau \mid \mathbb{M}\, I\, \tau \mid L^I\ \tau$
			$\alpha \mid \forall \alpha : K . \tau \mid \forall i : S . \tau \mid \lambda_t i : S . \tau \mid \tau\ I \mid \exists i : S . \tau \mid c \Rightarrow \tau \mid c \& \tau$
Constraints	c	$::=$	$I = I \mid I < I \mid c \wedge c$
Lin. context for term variables	Γ	$::=$	$. \mid \Gamma, x : \tau$
Unres. context for term variables	Ω	$::=$	$. \mid \Omega, x : \tau$
Unres. context for index variables	Θ	$::=$	$. \mid \Theta, i : S$
Unres. context for type variables	Ψ	$::=$	$. \mid \Psi, \alpha : K$

Definition 64 (Binary sum of multiplicity context).

$$\Omega_1 \oplus \Omega_2 \triangleq \begin{cases} \Omega_2 & \Omega_1 = . \\ (\Omega_1' \oplus \Omega_2), x : \tau & \Omega_1 = \Omega_1', x : \tau \wedge (x : -) \notin \Omega_2 \\ \text{undefined} & \Omega_1 = \Omega_1', x : \tau \wedge (x : \tau) \in \Omega_2 \end{cases}$$

Definition 65 (Binary sum of affine context).

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma_1' \oplus \Gamma_2), x : \tau & \Gamma_1 = \Gamma_1', x : \tau \wedge (x : -) \notin \Gamma_2 \\ \text{undefined} & \Gamma_1 = \Gamma_1', x : \tau \wedge (x : -) \in \Gamma_2 \end{cases}$$

2.2 Typesystem

$\boxed{\text{Typing } \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}$

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau}\text{ T-var1} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma \vdash x : \tau}\text{ T-var2}$$

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}}\text{ T-unit} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathsf{b}}\text{ T-base} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{nil} : L^0\ \tau}\text{ T-nil}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : L^n\tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1}\tau}\text{ T-cons}$$

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : L^n\ \tau \\ \Psi; \Theta; \Delta, n = 0; \Omega; \Gamma_2 \vdash e_1 : \tau' \qquad \Psi; \Theta; \Delta, n > 0; \Omega; \Gamma_2, h : \tau, t : L^{n-1}\tau \vdash e_2 : \tau'\end{array}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{match}\ e\ \mathsf{with}\ |\mathit{nil} \mapsto e_1\ |h :: t \mapsto e_2 : \tau'}\text{ T-match}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \qquad \Theta; \Delta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S\ .\tau}\text{ T-existI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s.\tau \qquad \Psi; \Theta, s; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau' \qquad \Psi; \Theta; \Delta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'}\text{ T-existE}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x.e : (\tau_1 \multimap \tau_2)}\text{ T-lam}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1\ e_2 : \tau_2}\text{ T-app}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Psi, \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'}\text{ T-sub}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Psi; \Theta; \Delta \models \Gamma' \sqsubseteq \Gamma \qquad \Psi; \Theta; \Delta \models \Omega' \sqsubseteq \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau}\text{ T-weaken}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \langle\!\langle e_1, e_2 \rangle\!\rangle : (\tau_1 \otimes \tau_2)}\text{ T-tensorI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\langle\!\langle x, y \rangle\!\rangle = e\ \mathsf{in}\ e' : \tau}\text{ T-tensorE}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1\ \&\ \tau_2)}\text{ T-withI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \,\&\, \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{fst}(e) : \tau_1} \text{ T-fst} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \,\&\, \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{snd}(e) : \tau_2} \text{ T-snd}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{inl}(e) : \tau_1 \oplus \tau_2} \text{ T-inl} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{inr}(e) : \tau_1 \oplus \tau_2} \text{ T-inr}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1; e_2 : \tau} \text{ T-case}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; . \vdash !e : !\tau} \text{ T-ExpI} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : !\tau \qquad \Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\,!\,x = e\ \mathsf{in}\ e' : \tau'} \text{ T-ExpE}$$

$$\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K\ .\tau)} \text{ T-tabs}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha : K\ .\tau) \qquad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\,[] : (\tau[\tau'/\alpha])} \text{ T-tapp}$$

$$\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S\ .\tau)} \text{ T-iabs} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S\ .\tau) \qquad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\,[] : (\tau[I/i])} \text{ T-iapp}$$

$$\frac{\Psi; \Theta; \Delta; \Omega, x : \tau; . \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; . \vdash \mathsf{fix} x.e : \tau} \text{ T-fix} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{ret}\, e : \mathbb{M}\, 0\, \tau} \text{ T-ret}$$

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \mathbb{M}\, n_1\, \tau_1 \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}\, n_2\, \tau_2 \qquad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+\end{array}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{bind}\, x = e_1\ \mathsf{in}\ e_2 : \mathbb{M}(I_1 + I_2)\, \tau_2} \text{ T-bind}$$

$$\frac{\Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^I : \mathbb{M}\, I\, \mathbf{1}} \text{ T-tick}$$

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : [I_1]\, \tau_1 \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(I_1 + I_2)\, \tau_2 \qquad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+\end{array}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{release}\, x = e_1\ \mathsf{in}\ e_2 : \mathbb{M}\, I_2\, \tau_2} \text{ T-release}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{store}\, e : \mathbb{M}\, I\, ([I]\, \tau)} \text{ T-store} \qquad \frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.\ e : (c \Rightarrow \tau)} \text{ T-CI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e[] : \tau} \text{ T-CE} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{ T-CAndI}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau) \qquad \Psi; \Theta; \Delta, c; \Omega; \Gamma, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{clet}\, x = e\ \mathsf{in}\ e' : \tau'} \text{ T-CAndE}$$

Figure 9: Typing rules for λ-Amor

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{ sub-refl} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1 \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau_1' \multimap \tau_2'} \text{ sub-arrow}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau_1' \otimes \tau_2'} \text{ sub-tensor}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \,\&\, \tau_2 <: \tau_1' \,\&\, \tau_2'} \text{ sub-with}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau_1' \oplus \tau_2'} \text{ sub-sum}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Theta; \Delta \models I' \leqslant I}{\Psi; \Theta; \Delta \vdash [I]\,\tau <: [I']\,\tau'} \text{ sub-potential}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Theta; \Delta \models I \leqslant I'}{\Psi; \Theta; \Delta \vdash \mathbb{M}\,I\,\tau <: \mathbb{M}\,I'\,\tau'} \text{ sub-monad} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash !\tau <: !\tau'} \text{ sub-Exp}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n\ \tau <: L^n\ \tau'} \text{ sub-list} \qquad \frac{\Psi; \Theta; \Delta, s \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s.\tau <: \exists s.\tau'} \text{ sub-exist}$$

$$\frac{\Psi, \alpha; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha.\tau_1 <: \forall \alpha.\tau_2} \text{ sub-typePoly} \qquad \frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i.\tau_1 <: \forall i.\tau_2} \text{ sub-indexPoly}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \qquad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \Rightarrow \tau_1 <: c_2 \Rightarrow \tau_2} \text{ sub-constraint}$$

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \qquad \Theta; \Delta \models c_1 \implies c_2}{\Psi; \Theta; \Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{ sub-CAnd}$$

$$\frac{\Psi; \Theta, i : S; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \lambda_t i : S\ .\tau <: \lambda_t i : S\ .\tau'} \text{ sub-familyAbs} \qquad \frac{\Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \lambda_t i : S\ .\tau\ I <: \tau[I/i]} \text{ sub-familyApp1}$$

$$\frac{\Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: \lambda_t i : S\ .\tau\ I} \text{ sub-familyApp2}$$

Figure 10: Subtyping

$$\frac{}{\Psi; \Theta; \Delta \vdash \Omega \sqsubseteq .}\ \text{sub-mBase}$$

$$\frac{x : \tau' \in \Omega_1 \qquad \Psi; \Theta; \Delta \vdash \tau' <: \tau \qquad \Psi; \Theta; \Delta \vdash \Omega_1/x \sqsubseteq \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 \sqsubseteq \Omega_2, x : \tau}\ \text{sub-mInd}$$

Figure 11: Ω Subtyping

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma \sqsubseteq .}\ \text{sub-lBase}$$

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta; \Delta \vdash \tau' <: \tau \qquad \Psi; \Theta; \Delta \vdash \Gamma_1/x \sqsubseteq \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2, x : \tau}\ \text{sub-lBase}$$

Figure 12: Γ Subtyping

$$\frac{}{\Theta, i : S; \Delta \vdash i : S}\ \text{S-var} \qquad \frac{}{\Theta; \Delta \vdash N : \mathbb{N}}\ \text{S-nat} \qquad \frac{}{\Theta; \Delta \vdash R : \mathbb{R}^+}\ \text{S-real} \qquad \frac{\Theta; \Delta \vdash i : \mathbb{N}}{\Theta; \Delta \vdash i : \mathbb{R}^+}\ \text{S-real1}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{N}}\ \text{S-add-Nat} \qquad \frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{R}^+}\ \text{S-add-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta \vdash I_2 : \mathbb{N} \qquad \Theta; \Delta \models I_1 \geqslant I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{N}}\ \text{S-minus-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \qquad \Theta; \Delta \vdash I_2 : \mathbb{R}^+ \qquad \Theta; \Delta \models I_1 \geqslant I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{R}^+}\ \text{S-minus-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{N}}\ \text{S-bSum-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \qquad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{R}^+}\ \text{S-bSum-Real} \qquad \frac{\Theta, i : S; \Delta \vdash I : S'}{\Theta; \Delta \vdash \lambda_s i.I : S \rightarrow S'}\ \text{S-family}$$

Figure 13: Typing rules for sorts

$$\frac{}{\Psi;\Theta;\Delta \vdash \mathbf{1} : Type}\ \text{K-unit} \qquad \frac{}{\Psi;\Theta;\Delta \vdash \mathsf{b} : Type}\ \text{K-base}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \vdash I : S}{\Psi;\Theta;\Delta \vdash L^{I}\tau : K}\ \text{K-List} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \multimap \tau_2 : K}\ \text{K-arrow}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \otimes \tau_2 : K}\ \text{K-tensor}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \;\&\; \tau_2 : K}\ \text{K-with} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau_1 : K \qquad \Psi;\Theta;\Delta \vdash \tau_2 : K}{\Psi;\Theta;\Delta \vdash \tau_1 \oplus \tau_2 : K}\ \text{K-or}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash\ !\tau : K}\ \text{K-Exp} \qquad \frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \vdash I : \mathbb{R}^{+}}{\Psi;\Theta;\Delta \vdash [I]\,\tau : K}\ \text{K-lab}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \vdash I : \mathbb{R}^{+}}{\Psi;\Theta;\Delta \vdash \mathbb{M}\, I\, \tau : K}\ \text{K-monad} \qquad \frac{\Psi, \alpha : K';\Theta;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \forall\alpha.\tau : K}\ \text{K-tabs}$$

$$\frac{\Psi;\Theta, i : S;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \forall i.\tau : K}\ \text{K-iabs} \qquad \frac{\Psi;\Theta;\Delta, c \vdash \tau : K}{\Psi;\Theta;\Delta \vdash c \Rightarrow \tau : K}\ \text{K-constraint}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : K \qquad \Theta;\Delta \models c}{\Psi;\Theta;\Delta \vdash c \& \tau : K}\ \text{K-consAnd} \qquad \frac{\Psi;\Theta, i : S;\Delta \vdash \tau : K}{\Psi;\Theta;\Delta \vdash \lambda_t i.\tau : S \to K}\ \text{K-family}$$

$$\frac{\Psi;\Theta;\Delta \vdash \tau : S \to K \qquad \Theta;\Delta \vdash I : S}{\Psi;\Theta;\Delta \vdash \tau\, I : K}\ \text{K-iapp}$$

Figure 14: Kind rules for types

2.3 Semantics

$\boxed{\text{Pure reduction, } e \Downarrow_t v}$ $\boxed{\text{Forcing reduction, } e \Downarrow^c_t v}$

$$\frac{e_1 \Downarrow_{t_1} v \qquad e_2 \Downarrow_{t_2} l}{e_1 :: e_2 \Downarrow_{t_1+t_2+1} v :: l}\ \text{E-cons} \qquad \frac{e_1 \Downarrow_{t_1} nil \qquad e_2 \Downarrow_{t_2} v}{\mathsf{match}\ e_1\ \mathsf{with}\ |nil \mapsto e_2\ |h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v}\ \text{E-matchNil}$$

$$\frac{e_1 \Downarrow_{t_1} v_h :: l \qquad e_3[v_h/h][l/t] \Downarrow_{t_2} v}{\mathsf{match}\ e_1\ \mathsf{with}\ |nil \mapsto e_2\ |h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v}\ \text{E-matchCons}$$

$$\frac{e_1 \Downarrow_{t_1} v \qquad e_2[v/x] \Downarrow_{t_2} v'}{e_1; x.e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-exist} \qquad \frac{e_1 \Downarrow_{t_1} \lambda x.e' \qquad e'[e_2/x] \Downarrow_{t_2} v'}{e_1\ e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-app}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2 \Downarrow_{t_2} v_2}{\langle\!\langle e_1, e_2\rangle\!\rangle \Downarrow_{t_1+t_2+1} \langle\!\langle v_1, v_2\rangle\!\rangle}\ \text{E-TI} \qquad \frac{e \Downarrow_{t_1} \langle\!\langle v_1, v_2\rangle\!\rangle \qquad e'[v_1/x][v_2/y] \Downarrow_{t_2} v}{\mathsf{let}\langle\!\langle x, y\rangle\!\rangle = e\ \mathsf{in}\ e' \Downarrow_{t_1+t_2+1} v}\ \text{E-TE}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2 \Downarrow_{t_2} v_2}{\langle e_1, e_2\rangle \Downarrow_{t_1+t_2+1} \langle v_1, v_2\rangle}\ \text{E-WI} \qquad \frac{e \Downarrow_t \langle v_1, v_2\rangle}{\mathsf{fst}(e) \Downarrow_{t+1} v_1}\ \text{E-fst} \qquad \frac{e \Downarrow_t \langle v_1, v_2\rangle}{\mathsf{fst}(e) \Downarrow_{t+1} v_2}\ \text{E-snd}$$

$$\frac{e \Downarrow_t v}{\mathsf{inl}(e) \Downarrow_{t+1} \mathsf{inl}(v)}\ \text{E-inl} \qquad \frac{e \Downarrow_t v}{\mathsf{inr}(e) \Downarrow_{t+1} \mathsf{inr}(v)}\ \text{E-inr} \qquad \frac{e \Downarrow_{t_1} \mathsf{inl}(v) \qquad e'[v/x] \Downarrow_{t_2} v'}{\mathsf{case}\ e\ \mathsf{of}\ e'; e'' \Downarrow_{t_1+t_2+1} \mathsf{inl}(v')}\ \text{E-case1}$$

$$\frac{e \Downarrow_{t_1} \mathsf{inr}(v) \qquad e''[v/y] \Downarrow_{t_2} v''}{\mathsf{case}\ e\ \mathsf{of}\ e'; e'' \Downarrow_{t_1+t_2+1} \mathsf{inl}(v'')}\ \text{E-case2} \qquad \frac{}{!e \Downarrow_0 !e}\ \text{E-expI}$$

$$\frac{e \Downarrow_{t_1} !e'' \qquad e'[e''/x] \Downarrow_{t_2} v}{\mathsf{let}\,!\,x = e\ \mathsf{in}\ e' \Downarrow_{t_1+t_2+1} v}\ \text{E-expE} \qquad \frac{e[\mathsf{fix}x.e/x] \Downarrow_t v}{\mathsf{fix}x.e \Downarrow_{t+1} v}\ \text{E-fix}$$

$$\frac{v \in \{(), x, nil, \lambda y.e, \Lambda.e, \mathsf{ret}\,e, \mathsf{bind}\,x = e_1\ \mathsf{in}\ e_2, \uparrow^\kappa, \mathsf{release}\,x = e_1\ \mathsf{in}\ e_2, \mathsf{store}\,e\}}{v \Downarrow_0 v}\ \text{E-val}$$

$$\frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_2} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-tapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_2} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-iapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \qquad e' \Downarrow_{t_1} v}{e\,[\,] \Downarrow_{t_1+t_2+1} v}\ \text{E-CE}$$

$$\frac{e_1 \Downarrow_{t_1} v \qquad e_2[v/x] \Downarrow_{t_2} v'}{\mathsf{clet}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow_{t_1+t_2+1} v'}\ \text{E-CandE} \qquad \frac{e \Downarrow_t v}{\mathsf{ret}\, e \Downarrow^0_{t+1} v}\ \text{E-return}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad v_1 \Downarrow^{c_1}_{t_2} v_1' \qquad e_2[v_1'/x] \Downarrow_{t_3} v_2 \qquad v_2 \Downarrow^{c_2}_{t_4} v_2'}{\mathsf{bind}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow^{c_1+c_2}_{t_1+t_2+t_3+t_4+1} v_2'}\ \text{E-bind} \qquad \frac{}{\uparrow^\kappa \Downarrow^\kappa_1 ()}\ \text{E-tick}$$

$$\frac{e_1 \Downarrow_{t_1} v_1 \qquad e_2[v_1/x] \Downarrow_{t_2} v_2 \qquad v_2 \Downarrow^{c}_{t_3} v_2'}{\mathsf{release}\, x = e_1 \ \mathsf{in}\ e_2 \Downarrow^{c}_{t_1+t_2+t_3+1} v_2'}\ \text{E-release} \qquad \frac{e \Downarrow_t v}{\mathsf{store}\, e \Downarrow^0_{t+1} v}\ \text{E-store}$$

Figure 15: Evaluation rules: pure and forcing

2.4 Model

Definition 66 (Value and expression relation)**.**

$$
\begin{array}{lll}
[\![\mathbf{1}]\!] & \triangleq & \{(p,T,())\} \\
[\![\mathsf{b}]\!] & \triangleq & \{(p,T,v) \mid v \in [\![\mathsf{b}]\!]\} \\
[\![L^0\tau]\!] & \triangleq & \{(p,T,nil)\} \\
[\![L^{s+1}\tau]\!] & \triangleq & \{(p,T,v::l) | \exists p_1,p_2.p_1+p_2 \leqslant p \wedge (p_1,T,v) \in [\![\tau]\!] \wedge (p_2,T,l) \in [\![L^s\tau]\!]\} \\
[\![\tau_1 \otimes \tau_2]\!] & \triangleq & \{(p,T,\langle\!\langle v_1,v_2\rangle\!\rangle) \mid \exists p_1,p_2.p_1+p_2 \leqslant p \wedge (p_1,T,v_1) \in [\![\tau_1]\!] \wedge (p_2,T,v_2) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \mathbin{\&} \tau_2]\!] & \triangleq & \{(p,T,\langle v_1,v_2\rangle) \mid (p,T,v_1) \in [\![\tau_1]\!] \wedge (p,T,v_2) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \oplus \tau_2]\!] & \triangleq & \{(p,T,\mathsf{inl}(v)) \mid (p,T,v) \in [\![\tau_1]\!]\} \cup \{(p,T,\mathsf{inr}(v)) \mid (p,T,v) \in [\![\tau_2]\!]\} \\
[\![\tau_1 \multimap \tau_2]\!] & \triangleq & \{(p,T,\lambda x.e) \mid \forall p',e',T'{<}T\ .(p',T',e') \in [\![\tau_1]\!]_{\mathcal{E}} \implies (p+p',T',e[e'/x]) \in [\![\tau_2]\!]_{\mathcal{E}}\} \\
[\![!\tau]\!] & \triangleq & \{(p,T,!e) \mid (0,T,e) \in [\![\tau]\!]_{\mathcal{E}}\} \\
[\![[n]\,\tau]\!] & \triangleq & \{(p,T,v) \mid \exists p'.p'+n \leqslant p \wedge (p',T,v) \in [\![\tau]\!]\}\} \\
[\![\mathbb{M}\,n\,\tau]\!] & \triangleq & \{(p,T,v) \mid \forall n',v',T'{<}T\ .v \Downarrow_{T'}^{n'} v' \implies \exists p'.n'+p' \leqslant p+n \wedge (p',T-T',v') \in [\![\tau]\!]\} \\
[\![\forall\alpha.\tau]\!] & \triangleq & \{(p,T,\Lambda.e) \mid \forall\tau',T'{<}T\ .(p,T',e) \in [\![\tau[\tau'/\alpha]]\!]_{\mathcal{E}}\} \\
[\![\forall i.\tau]\!] & \triangleq & \{(p,T,\Lambda.e) \mid \forall I,T'{<}T\ .(p,T',e) \in [\![\tau[I/i]]\!]_{\mathcal{E}}\} \\
[\![c \Rightarrow \tau]\!] & \triangleq & \{(p,T,\Lambda.e) \mid .\models c \implies (p,T,e) \in [\![\tau]\!]_{\mathcal{E}}\} \\
[\![c\&\tau]\!] & \triangleq & \{(p,T,v) \mid .\models c \wedge (p,T,v) \in [\![\tau]\!]\} \\
[\![\exists s.\tau]\!] & \triangleq & \{(p,T,v) \mid \exists s'.(p,T,v) \in [\![\tau[s'/s]]\!]\} \\
[\![\lambda_t i.\tau]\!] & \triangleq & f \text{ where } \forall I.\ f\ I = [\![\tau[I/i]]\!] \\
[\![\tau\ I]\!] & \triangleq & [\![\tau]\!]\ I \\
\\
[\![\tau]\!]_{\mathcal{E}} & \triangleq & \{(p,T,e) \mid \forall\ T'{<}T, v.e \Downarrow_{T'} v \implies (p,T-T',v) \in [\![\tau]\!]\}
\end{array}
$$

Definition 67 (Interpretation of typing contexts)**.**

$$
\begin{array}{lll}
[\![\Gamma]\!]_{\mathcal{E}} & = & \{(p,T,\gamma) \mid \exists f : \mathcal{V}ars \to \mathcal{P}ots. \\
& & (\forall x \in dom(\Gamma).\ (f(x),T,\gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}}) \wedge (\sum_{x \in dom(\Gamma)} f(x) \leqslant p)\} \\
[\![\Omega]\!]_{\mathcal{E}} & = & \{(0,T,\delta) \mid (\forall x \in dom(\Omega).(0,T,\delta(x)) \in [\![\tau]\!]_{\mathcal{E}})\}
\end{array}
$$

Definition 68 (Type and index substitutions)**.** $\sigma : TypeVar \to Type$, $\iota : IndexVar \to Index$

Lemma 69 (Value monotonicity lemma)**.** $\forall p,p',v,\tau.$
$(p,T,v) \in [\![\tau]\!] \wedge p \leqslant p' \wedge T' \leqslant T \implies (p',T',v) \in [\![\tau]\!]$

Proof. Proof by induction on τ ☐

Lemma 70 (Expression monotonicity lemma)**.** $\forall p,p',v,\tau.$
$(p,T,e) \in [\![\tau]\!]_{\mathcal{E}} \wedge p \leqslant p' \wedge T' \leqslant T \implies (p',T',e) \in [\![\tau]\!]_{\mathcal{E}}$

Proof. From Definition 66 and Lemma 69 ☐

Theorem 71 (Fundamental theorem)**.** $\forall\Theta,\Omega,\Gamma,e,\tau,T,p_l,\gamma,\delta,\sigma,\iota.$
$\Psi;\Theta;\Delta;\Omega;\Gamma \vdash e : \tau \wedge (p_l,T,\gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}} \wedge (0,T,\delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}} \wedge .\models \Delta\ \iota \implies$
$(p_l,T,e\ \gamma\delta) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$.

Proof. Proof by induction on the typing judgment

1. T-var1:

$$\frac{}{\Psi;\Theta;\Delta;\Omega;\Gamma,x:\tau \vdash x:\tau}\ \text{T-var1}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, x : \tau\ \sigma\iota]\!]_{\mathcal{E}}$ and $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in [\![\Gamma, x : \tau\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Definition 67 we know that $\exists f.(f(x), T, \gamma(x)) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ where $f(x) \leqslant p_l$

Therefore from Lemma 70 we get $(p_l, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

2. T-var2:

$$\frac{}{\Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma \vdash x : \tau} \text{ T-var2}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$ and $(0, T, \delta) \in [\![(\Omega, x : \tau)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(0, T, \delta) \in [\![(\Omega, x : \tau)\ \sigma\iota]\!]_{\mathcal{E}}$ therefore from Definition 67 we know that
$(0, T, \delta(x)) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Lemma 70 we get $(p_l, T, x\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

3. T-unit:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{ T-unit}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, ()\ \delta\gamma) \in [\![\mathbf{1}\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall\ T'{<}T, v'.() \Downarrow_{T'} v' \implies (p_l, T - T', v') \in [\![\mathbf{1}]\!]$

This means given some $T'{<}T, v'$ s.t $() \Downarrow_{T'} v'$ it suffices to prove that
$(p_l, T - T', v') \in [\![\mathbf{1}]\!]$

From (E-val) we know that $T'{=}\ 0$ and $v' = ()$, therefore it suffices to prove that
$(p_l, T, ()) \in [\![\mathbf{1}]\!]$
We get this directly from Definition 66

4. T-base:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathsf{b}} \text{ T-base}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, c) \in [\![\mathsf{b}]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall\, T'{<}T, v'.c \Downarrow_{T'} v' \implies (p_l, T - T', v') \in [\![\mathbf{1}]\!]$

This means given some $T'{<}T, v'$ s.t $c \Downarrow_{T'} v'$ it suffices to prove that $(p_l, T - T', v') \in [\![\mathbf{1}]\!]$

From (E-val) we know that $T'{=}\ 0$ and $v' = c$, therefore it suffices to prove that $(p_l, T, c) \in [\![\mathsf{b}]\!]$
We get this directly from Definition 66

5. T-nil:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash nil : L^0\ \tau}\text{ T-nil}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, nil\ \delta\gamma) \in [\![L^0\ \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall\, T'{<}T, v'.nil \Downarrow_{T'} v' \implies (p_l, T - T', v') \in [\![L^0\ \tau\ \sigma\iota]\!]$

This means given some $T'{<}T, v'$ s.t $nil \Downarrow_{T'} v'$ it suffices to prove that $(p_l, T - T', v') \in [\![L^0\ \tau\ \sigma\iota]\!]$

From (E-val) we know that $T'{=}\ 0$ and $v' = nil$, therefore it suffices to prove that $(p_l, T, nil) \in [\![L^0\ \tau\ \sigma\iota]\!]$
We get this directly from Definition 66

6. T-cons:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : L^n\tau \qquad \Theta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1}\tau}\text{ T-cons}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, (e_1 :: e_2)\ \delta\gamma) \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t\ {<}T, v'.(e_1 :: e_2)\ \delta\gamma \Downarrow_t v' \implies (p_l, T\ {-}t, v') \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]$

This means given some $t\ {<}T, v'$ s.t $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v'$, it suffices to prove that $(p_l, T\ {-}t, v') \in [\![L^{n+1}\ \tau\ \sigma\iota]\!]$

From (E-cons) we know that $\exists v_f, l.v' = v_f :: l$

Therefore from Definition 66 it suffices to prove that

$$\exists p_1, p_2.p_1 + p_2 \leqslant p_l \wedge (p_1, T\ {-}t, v_f) \in [\![\tau\ \sigma\iota]\!] \wedge (p_2, T\ {-}t, l) \in [\![L^n\tau\ \sigma\iota]\!] \qquad \text{(F-C0)}$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t $(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

IH1:

$(p_{l1}, T, e_1\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t1 < T. e_1\ \delta\gamma \Downarrow_{t1} v_f \implies (p_{l1}, T - t1, v_f) \in [\![\tau]\!]$

Since we are given that $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t1 < t.\ e_1\ \delta\gamma \Downarrow_{t1} v_f$

Since $t1 < t < T$, therefore we have $(p_{l1}, T - t1, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-C1)

IH2:

$(p_{l2}, T, e_2\ \delta\gamma) \in [\![L^n\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t2 < T .e_2\ \delta\gamma \Downarrow_{t2} l \implies (p_{l2}, T - t2, l \in [\![L^n\tau\ \sigma\iota]\!]$

Since we are given that $(e_1 :: e_2)\ \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t2 < t - t1.\ e_2\ \delta\gamma \Downarrow_{t2} l$

Since $t2 < t - t1 < t < T$, therefore we have

$(p_{l2}, T - t2, l) \in [\![L^n\tau\ \sigma\iota]\!]$ (F-C2)

In order to prove (F-C0) we choose p_1 as p_{l1} and p_2 as p_{l2} and it suffices to prove that

$(p_{l1}, T - t, v) \in [\![\tau\ \sigma\iota]\!] \land (p_{l2}, T - t, l) \in [\![L^n\tau\ \sigma\iota]\!]$

Since $t = t_1 + t_2 + 1$ therefore from (F-C1) and Lemma 69 we get $(p_{l1}, T - t, v) \in [\![\tau\ \sigma\iota]\!]$

Similarly from (F-C2) and Lemma 69 we also get $(p_{l2}, T - t, l) \in [\![L^n\tau\ \sigma\iota]\!]$

7. T-match:

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : L^n\ \tau \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_1 : \tau' \qquad \Psi; \Theta; \Delta, n > 0; \Omega; \Gamma_2, h : \tau, t : L^{n-1}\tau \vdash e_2 : \tau'\end{array}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2 : \tau'} \text{ T-match}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, (\mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{match}\ e\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-M0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

IH1

$(p_{l1}, T, e\ \delta\gamma) \in [\![L^n\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t' < T . e\ \delta\gamma \Downarrow_{t'} v_1 \implies (p_{l1}, T - t', v_1) \in [\![L^n\tau\ \sigma\iota]\!]$

Since we know that (match e with $|nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t' < t, v_1.e\ \delta\gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have $(p_{l1}, T - t', v_1) \in [\![L^n\tau\ \sigma\iota]\!]$

2 cases arise:

(a) $v_1 = nil$:
In this case we know that $n = 0$ therefore
<u>IH2</u>
$(p_{l2}, T, e_1\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_1 < T . e_1\ \delta\gamma \Downarrow_{t_1} v_f \implies (p_{l2}, T - t_1, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that (match e with $|nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_1 < t.\ e_1\ \delta\gamma \Downarrow_{t_1} v_f$.
Since $t_1 < t < T$ therefore we have
$(p_{l2}, T - t_1, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And from Lemma 69 we get
$(p_{l2} + p_{l1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get
$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
And we are done

(b) $v_1 = v :: l$:
In this case we know that $n > 0$
<u>IH2</u>
$(p_{l2} + p_{l1}, T, e_2\ \delta\gamma') \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
where
$\gamma' = \gamma \cup \{h \mapsto v\} \cup \{t \mapsto l\}$

This means from Definition 66 we have
$\forall t_2 < T . e_2\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{l1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that (match e with $|nil \mapsto e_1\ |h :: t \mapsto e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_2 < t.\ e_2\ \delta\gamma' \Downarrow_{t_2} v_f$.
Since $t_2 < t < T$ therefore we have
$(p_{l2} + p_{l1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

From Lemma 69 we get
$(p_{l2} + p_{l1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get
$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
And we are done

8. T-existI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \qquad \Theta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S\, .\tau} \text{ T-existI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, e\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.e\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $e\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\exists s.\tau\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$\exists s'.(p_l, T - t, v_f) \in [\![\tau[s'/s]\ \sigma\iota]\!]$ (F-E0)

<u>IH</u>: $(p_l, T, e\ \delta\gamma) \in [\![\tau[n/s]\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t' < T .e\ \delta\gamma \Downarrow_{t'} v_f \implies (p_l, T - t', v_f) \in [\![\tau[n/s]\ \sigma\iota]\!]$

Since we are given that $e\ \delta\gamma \Downarrow_t v_f$ therefore we get
$(p_l, T - t, v_f) \in [\![\tau[n/s]\ \sigma\iota]\!]$ (F-E1)

To prove (F-E0) we choose s' as n and we get the desired from (F-E1)

9. T-existsE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s.\tau \qquad \Psi; \Theta, s; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau' \qquad \Theta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{ T-existE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, (e; x.e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.(e; x.e')\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given soem $t < T, v_f$ s.t $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-EE0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t
$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>
$(p_{l1}, T, e\ \delta\gamma) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T . e\ \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in [\![\exists s.\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we know that $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_1 < t, v_1.e\ \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have

$(p_{l1}, T - t_1, v_1) \in [\![\exists s.\tau\ \sigma\iota]\!]$

Therefore from Definition 66 we have

$\exists s'.(p_{l1}, T - t_1, v_1) \in [\![\tau[s'/s]\ \sigma\iota]\!]$ (F-EE1)

<u>IH2</u>

$(p_{l1} + p_{l2}, T, e'\ \delta'\gamma) \in [\![\tau'\ \sigma\iota']\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto e_1\}$ and $\iota' = \iota \cup \{s \mapsto s'\}$

This means from Definition 66 we have

$\forall t_2 < T . e'\ \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l1} + p_{l2}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota']\!]$

Since we know that $(e; x.e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_2 < t.\ e'\ \delta'\gamma \Downarrow_{t_2} v_f$.

Since $t_2 < t < T$ therefore we have

$(p_{l1} + p_{l2}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota']\!]$

Since $p_l = p_{l1} + p_{l2}$ therefore we get

$(p_l, T - t_2, v_f) \in [\![\tau'\ \sigma\iota']\!]$

From Lemma 69 we get

$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota']\!]$

And finally since we have $\Psi; \Theta \vdash \tau'$ therefore we also have

$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And we are done

10. T-lam:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x.e : (\tau_1 \multimap \tau_2)} \text{ T-lam}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, (\lambda x.e)\ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\lambda x.e)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\lambda x.e)\ \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\lambda x.e)\ \delta\gamma$

Therefore it suffices to prove that

$(p_l, T, (\lambda x.e)\ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$\forall p', e', T'{<}T\ .(p', T', e') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p_l + p', T', e[e'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $p', e', T'{<}T$ s.t $(p', T', e') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that
$(p_l + p', T', e[e'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-L1)

From IH we know that
$(p_l + p', T, e\ \delta\gamma') \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$
where
$\gamma' = \gamma \cup \{x \mapsto e'\}$

Therefore from Lemma 70 we get the desired

11. T-app:

$$\frac{\Psi;\Theta;\Delta;\Omega_1;\Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \qquad \Psi;\Theta;\Delta;\Omega_2;\Gamma_2 \vdash e_2 : \tau_1}{\Psi;\Theta;\Delta;\Omega_1 \oplus \Omega_2;\Gamma_1 \oplus \Gamma_2 \vdash e_1\ e_2 : \tau_2}\ \text{T-app}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, e_1\ e_2\ \delta\gamma) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau_2\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-A0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t
$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>
$(p_{l1}, T, e_1\ \delta\gamma) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_1 < T\ .e_1 \Downarrow_{t_1} \lambda x.e \implies (p_{l1}, T - t_1, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$
Since we know that $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_1 < t.e_1 \Downarrow_{t_1} \lambda x.e$, therefore we have
$(p_{l1}, T - t_1, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$

Therefore from Definition 66 we have
$\forall p', e_1, T_1{<}T - t_1.(p', T_1, e'_1) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p_{l1} + p', T_1, e[e'_1/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-A1)

<u>IH2</u>
$(p_{l2}, T - t_1 - 1, e_2\ \delta\gamma) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$ (F-A2)

Instantiating (F-A1) with p_{l2}, $e_2\ \delta\gamma$ and $T - t_1 - 1$ we get
$(p_{l1} + p_{l2}, T - t_1 - 1, e[e_2\ \delta\gamma/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_2 < T - t_1 - 1. e[e_2\ \delta\gamma/x] \Downarrow_{t_2} v_f \implies (p_{l1} + p_{l2}, T - t_1 - 1 - t_2, v_f) \in [\![\tau_2\ \sigma\iota]\!]$

Since we know that $(e_1\ e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_2. e[e_2\ \delta\gamma/x] \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$, therefore we have
$(p_{l1} + p_{l2}, T - t_1 - t_2 - 1, v_f) \in [\![\tau_2\ \sigma\iota]\!]$ where $p_{l1} + p_{l2} = p_l$
Since from E-app we know that $t = t_1 + t_2 + 1$, therefore we have proved (F-A0)

12. T-sub:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{ T-sub}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, e\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$
<u>IH</u> $(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH and Lemma 73

13. T-weaken:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Psi; \Theta; \Delta \models \Gamma' <: \Gamma \qquad \Psi; \Theta; \Delta \models \Omega' <: \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau} \text{ T-weaken}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma')\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega')\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in [\![(\Gamma')\sigma\iota]\!]_{\mathcal{E}}$ therefore from Lemma 74 we also have $(p_l, T, \gamma) \in [\![(\Gamma)\sigma\iota]\!]_{\mathcal{E}}$

Similarly since we are given that $(0, T, \delta) \in [\![(\Omega')\sigma\iota]\!]_{\mathcal{E}}$ therefore from Lemma 76 we also have $(0, T, \delta) \in [\![(\Omega)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u>:
$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH

14. T-tensorI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \langle\!\langle e_1, e_2 \rangle\!\rangle : (\tau_1 \otimes \tau_2)} \text{ T-tensorI}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, \langle\!\langle e_1, e_2 \rangle\!\rangle\ \delta\gamma) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T . \langle\langle e_1, e_2 \rangle\rangle \; \delta\gamma \Downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle \implies (p_l, T - t, \langle\langle v_{f1}, v_{f2} \rangle\rangle \in [\![(\tau_1 \otimes \tau_2) \; \sigma\iota]\!]$

This means given some $t < T$ s.t $\langle\langle e_1, e_2 \rangle\rangle \; \delta\gamma \Downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ it suffices to prove that

$(p_l, T - t, \langle\langle v_{f1}, v_{f2} \rangle\rangle) \in [\![(\tau_1 \otimes \tau_2) \; \sigma\iota]\!]$ (F-TI0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>:

$(p_{l1}, T, e_1 \; \delta\gamma) \in [\![\tau_1 \; \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_1 < T . e_1 \; \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_{l1}, T - t_1, v_{f1}) \in [\![\tau_1 \; \sigma\iota]\!]$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \; \delta\gamma \Downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ therefore fom E-TI we know that $\exists t_1 < t . e_1 \; \delta\gamma \Downarrow_{t_1} v_{f1}$

Hence we have $(p_{l1}, T - t_1, v_{f1}) \in [\![\tau_1 \; \sigma\iota]\!]$ (F-TI1)

<u>IH2</u>:

$(p_{l2}, T, e_2 \; \delta\gamma) \in [\![\tau_2 \; \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_2 < T . e_2 \; \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_{l2}, T - t_2, v_{f2} \in [\![\tau_2 \; \sigma\iota]\!]$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \; \delta\gamma \Downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ therefore fom E-TI we also know that $\exists t_2 < t . e_2 \; \delta\gamma \Downarrow_{t_2} v_{f2}$ s.t

Since $t_2 < t < T$ therefore we have

$(p_{l2}, T - t_2, v_{f2}) \in [\![\tau_2 \; \sigma\iota]\!]$ (F-TI2)

Applying Lemma 69 on (F-TI1) and (F-TI2) and by using Definition 66 we get the desired.

15. T-tensorE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\langle\langle x, y \rangle\rangle = e \; \mathsf{in} \; e' : \tau} \text{ T-tensorE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2) \; \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega \; \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, (\mathsf{let}\langle\langle x, y \rangle\rangle = e \; \mathsf{in} \; e') \; \delta\gamma) \in [\![\tau \; \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f . (\mathsf{let}\langle\langle x, y \rangle\rangle = e \; \mathsf{in} \; e') \; \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau \; \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{let}\langle\langle x, y \rangle\rangle = e \; \mathsf{in} \; e') \; \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in [\![\tau \; \sigma\iota]\!]$ (F-TE0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1}, T, e\ \delta\gamma) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .e\ \delta\gamma \Downarrow_{t_1} \langle\!\langle v_1, v_2 \rangle\!\rangle\ \delta\gamma \implies (p_{l1}, T - t_1, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\langle\!\langle x, y \rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_1 < t, v_1, v_2.e\ \delta\gamma \Downarrow_{t_1} \langle\!\langle v_1, v_2 \rangle\!\rangle$. Therefore we have

$(p_{l1}, T - t_1, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 we know that

$\exists p_1, p_2.p_1 + p_2 \leqslant p_{l1} \wedge (p_1, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p_2, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-TE1)

<u>IH2</u>

$(p_{l2} + p_1 + p_2, T, e'\ \delta\gamma') \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v_1\} \cup \{y \mapsto v_2\}$

This means from Definition 66 we have

$\forall t_2 < T\ .e'\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_1 + p_2, T - t_2, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\langle\!\langle x, y \rangle\!\rangle = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_2 < t.e'\ \delta\gamma' \Downarrow_{t_2} v_f$. Therefore we have

$(p_{l2} + p_1 + p_2, T - t_2, v_f) \in [\![\tau\ \sigma\iota]\!]$

From Lemma 69 we get

$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

And we are done

16. T-withI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \qquad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \ \&\ \tau_2)} \text{T-withI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, \langle e_1, e_2 \rangle\ \delta\gamma) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T\ .\langle e_1, e_2 \rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle \implies (p_l, T - t, \langle v_{f1}, v_{f2} \rangle) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]$

This means given $\langle e_1, e_2 \rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ it suffices to prove that

$(p_l, T - t, \langle v_{f1}, v_{f2} \rangle) \in [\![(\tau_1 \ \&\ \tau_2)\ \sigma\iota]\!]$ (F-WI0)

<u>IH1</u>:

$(p_l, T, e_1\ \delta\gamma) \in \llbracket \tau_1\ \sigma\iota \rrbracket_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_1 < T\ .e_1\ \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_l, T - t_1, v_{f1}) \in \llbracket \tau_1\ \sigma\iota \rrbracket$

Since we are given that $\langle e_1, e_2 \rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ therefore fom E-WI we know that $\exists t_1 < t.e_1\ \delta\gamma \Downarrow_{t_1} v_{f1}$

Since $t_1 < t < T$, therefore we have

$(p_l, T - t_1, v_{f1}) \in \llbracket \tau_1\ \sigma\iota \rrbracket$ (F-WI1)

<u>IH2</u>:

$(p_l, T, e_2\ \delta\gamma) \in \llbracket \tau_2\ \sigma\iota \rrbracket_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_2 < T\ .e_2\ \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_l, T - t_2, v_{f2} \in \llbracket \tau_2\ \sigma\iota \rrbracket$

Since we are given that $\langle e_1, e_2 \rangle\ \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ therefore fom E-WI we also know that $\exists t_2 < t.e_2\ \delta\gamma \Downarrow_{t_2} v_{f2}$

Since $t_2 < t < T$, therefore we have

$(p_l, T - t_2, v_{f2}) \in \llbracket \tau_2\ \sigma\iota \rrbracket$ (F-WI2)

Applying Lemma 69 on (F-W1) and (F-W2) we get the desired.

17. T-fst:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1\ \&\ \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{fst}(e) : \tau_1} \text{ T-fst}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma)\ \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega\ \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\mathsf{fst}(e))\ \delta\gamma) \in \llbracket \tau_1\ \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{fst}(e))\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau_1\ \sigma\iota \rrbracket$

This means given some $t < T, v_f$ s.t $(\mathsf{fst}(e))\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in \llbracket \tau_1\ \sigma\iota \rrbracket$ (F-F0)

<u>IH</u>

$(p_l, T, e\ \delta\gamma) \in \llbracket (\tau_1\ \&\ \tau_2)\ \sigma\iota \rrbracket_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .e\ \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle\ \delta\gamma \implies (p_l, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1\ \&\ \tau_2)\ \sigma\iota \rrbracket$

Since we know that $(\mathsf{fst}(e))\ \delta\gamma \Downarrow_t v_f$ therefore from E-fst we know that $\exists t_1 < t.v_1, v_2.e\ \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle$.

Since $t_1 < t < T$, therefore we have

$(p_l, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1\ \&\ \tau_2)\ \sigma\iota \rrbracket$

From Definition 66 we know that

$(p_l, T - t_1, v_1) \in [\![\tau_1\ \sigma\iota]\!]$

Finally using Lemma 69 we also have

$(p_l, T - t, v_1) \in [\![\tau_1\ \sigma\iota]\!]$

Since from E-fst we know that $v_f = v_1$, therefore we are done.

18. T-snd:

    Similar reasoning as in T-fst case above.

19. T-inl:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{inl}(e) : \tau_1 \oplus \tau_2}\ \text{T-inl}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, \mathsf{inl}(e)\ \delta\gamma) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T . \mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v) \implies (p_l, T - t, \mathsf{inl}(v) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$

This means given some $t < T$ s.t $\mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v)$ it suffices to prove that

$(p_l, T - t, \mathsf{inl}(v)) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$ (F-IL0)

<u>IH</u>:

$(p_l, T, e_1\ \delta\gamma) \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 66 we have

$\forall t_1 < T . e_1\ \delta\gamma \Downarrow_{t_1} v_{f1} \implies (p_l, T - t_1, v_{f1}) \in [\![\tau_1\ \sigma\iota]\!]$

Since we are given that $\mathsf{inl}(e)\ \delta\gamma \Downarrow_t \mathsf{inl}(v)$ therefore fom E-inl we know that $\exists t_1 < t . e\ \delta\gamma \Downarrow_{t_1} v$

Hence we have $(p_l, T - t_1, v) \in [\![\tau_1\ \sigma\iota]\!]$

From Lemma 69 we get $(p_l, T - t, v) \in [\![\tau_1\ \sigma\iota]\!]$

And finally from Definition 66 we get (F-IL0)

20. T-inr:

    Similar reasoning as in T-inr case above.

21. T-case:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1; e_2 : \tau}\ \text{T-case}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, (\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-C0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1}, T, e\ \delta\gamma) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t' < T\ .e\ \delta\gamma \Downarrow_{t'} v_1\ \delta\gamma \implies (p_{l1}, T - t', v_1) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$

Since we know that $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t' < t, v_1.e\ \delta\gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have

$(p_{l1}, T - t', v_1) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$

2 cases arise:

(a) $v_1 = \mathsf{inl}(v)$:

<u>IH2</u>

$(p_{l2} + p_{l1}, T - t', e_1\ \delta\gamma') \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v\}$

This means from Definition 66 we have

$\forall t_1 < T - t'.e_1\ \delta\gamma' \Downarrow_{t_1} v_f \implies (p_{l2}, T - t' - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{case}\ e\ \mathsf{of}\ e_1; e_2)\ \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t_1.e_1\ \delta\gamma' \Downarrow v_f$ where $t_1 = t - t' - 1$.

Since $t_1 = t - t' - 1 < T - t'$ therefore we have

$(p_{l2}, T - t' - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

From Lemma 69 we get

$(p_{l2} + p_{l1}, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

And we are done

(b) $v_1 = \mathsf{inr}(v)$:

Similar reasoning as in the inl case above.

22. T-ExpI:

$$\frac{\Psi; \Theta; \Delta; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; . \vdash !e :!\tau} \text{ T-ExpI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, !e\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T .(!e)\ \delta\gamma \Downarrow_t (!e)\ \delta\gamma \implies (p_l, T - t, (!e)\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]$

This means given some $t < T$ s.t $(!e)\ \delta\gamma \Downarrow_t (!e)\ \delta\gamma$ it suffices to prove that
$(p_l, T - t, (!e)\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$(0, T - t, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u>: $(0, T - t, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH

23. T-ExpE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e :!\tau \qquad \Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{let}\,!\,x = e \mathsf{\ in\ } e' : \tau'} \text{ T-ExpE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, (\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-E0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t
$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>
$(p_{l1}, T, e\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_1 < T .e\ \delta\gamma \Downarrow_{t_1} !e_1\ \delta\gamma \implies (p_{l1}, T - t_1, !e_1\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from (E-ExpE) we know that $\exists t_1 < t, e_1.e\ \delta\gamma \Downarrow_{t_1} !e_1\ \delta\gamma$.
Since $t_1 < t < T$, therefore we have

$(p_{l1}, T - t_1, !e_1\ \delta\gamma) \in [\![!\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$(0, T - t_1, e_1\ \delta\gamma) \in [\![\tau]\!]_{\mathcal{E}}$ (F-E1)

<u>IH2</u>

$(p_{l2}, T - t_1, e'\ \delta'\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\delta' = \delta \cup \{x \mapsto e_1\}$

This means from Definition 66 we have

$\forall t_2 < T - t_1 . e'\ \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l2}, T - t_1 - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that $(\mathsf{let}\,!\,x = e \mathsf{\ in\ } e')\ \delta\gamma \Downarrow_t v_f$ therefore from (E-ExpE) we know that $\exists t_2 . e'\ \delta'\gamma \Downarrow v_f$ where $t_2 = t - t_1 - 1$.

Since $t_2 = t - t_1 - 1 < T - t_1$, therefore we have

$(p_{l2}, T - t_1 - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

From Lemma 70 we get

$(p_{l2} + p_{l1}, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And we are done

24. T-tabs:

$$\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K . \tau)} \text{ T-tabs}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, (\Lambda.e)\ \delta\gamma) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f . (\Lambda.e)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\Lambda.e)\ \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\Lambda.e)\ \delta\gamma$

Therefore it suffices to prove that

$(p_l, T, (\Lambda.e)\ \delta\gamma) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that

$\forall \tau', T' < T . (p_l, T', e) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $\tau', T' < T$ it suffices to prove that

$(p_l, T', e) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-TAB0)

From IH we know that

$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma'\iota]\!]_{\mathcal{E}}$

where

$\sigma' = \gamma \cup \{\alpha \mapsto \tau'\}$

Therefore from Lemma 70 we get the desired

25. T-tapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall\alpha.\tau) \qquad \Psi; \Theta\Delta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\ [] : (\tau[\tau'/\alpha])} \text{ T-tapp}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, e\ []\ \delta\gamma) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$ (F-A0)

<u>IH</u>

$(p_l, T, e\ \delta\gamma) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .e \Downarrow_{t_1} \Lambda.e \implies (p_l, T - t_1, \Lambda.e) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t.e \Downarrow_{t_1} \Lambda.e$, therefore we have

$(p_l, T - t_1, \Lambda.e) \in [\![(\forall\alpha.\tau)\ \sigma\iota]\!]$

Therefore from Definition 66 we have

$\forall\tau'', T_1 < T - t_1.(p_l, T_1, e) \in [\![\tau[\tau''/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-A1)

Instantiating (F-A1) with the given τ' and $T - t_1 - 1$ we get

$(p_l, T - t_1 - 1, e) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 we have

$\forall t_2 < T - t_1 - 1.e \Downarrow_{t_2} v_f \implies (p_l, T - t_1 - t_2 - 1, v_f) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_2.e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$(p_l, T - t_1 - t_2 - 1, v_f) \in [\![\tau[\tau'/\alpha]\ \sigma\iota]\!]$ and we are done.

26. T-iabs:

$$\frac{\Psi; \Theta, i :S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i :S .\tau)} \text{ T-iabs}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma, \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, (\Lambda.e)\ \delta\gamma) \in [\![(\forall i.\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.(\Lambda.e)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![(\forall i.\tau)\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(\Lambda.e)\ \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\Lambda.e)\ \delta\gamma$
Therefore it suffices to prove that
$(p_l, T, (\Lambda.e)\ \delta\gamma) \in [\![(\forall i.\tau)\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$\forall I, T' < T .(p_l, T', e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $I, T' < T$ it suffices to prove that
$(p_l, T', e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-IAB0)

From IH we know that
$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota']\!]_{\mathcal{E}}$
where
$\iota' = \gamma \cup \{i \mapsto I\}$

Therefore from Lemma 70 we get the desired

27. T-iapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i :S .\tau) \qquad \Psi; \Theta; \Delta \vdash I :S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e\ [] : (\tau[I/i])} \text{ T-iapp}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, e\ []\ \delta\gamma) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v_f.(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau[I/i]\ \sigma\iota]\!]$

This means given some $t < T, v_f$ s.t $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau[I/i]\ \sigma\iota]\!]$ (F-A0)

<u>IH</u>
$(p_l, T, e\ \delta\gamma) \in [\![(\forall i.\tau)\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .e \Downarrow_{t_1} \Lambda.e \implies (p_l, T - t_1, \Lambda.e) \in [\![(\forall i.\tau)\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t.e \Downarrow_{t_1} \Lambda.e$, therefore we have

$(p_l, T - t_1, \Lambda.e) \in [\![(\forall i.\tau)\ \sigma\iota]\!]$

Therefore from Definition 66 we have

$\forall I, T_1 < T - t_1.(p_l, T_1, e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$ (F-IAP1)

Instantiating (F-IAP1) with the given I and $T - t_1 - 1$ we get

$(p_l, T - t_1 - 1, e) \in [\![\tau[I/i]\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 we have

$\forall t_2 < T - t_1 - 1.e \Downarrow_{t_2} v_f \implies (p_l, T - t_1 - t_2 - 1, v_f) \in [\![\tau[I/i]\ \sigma\iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-iapp we know that $\exists t_2.e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$(p_l, T - t_1 - t_2 - 1, v_f) \in [\![\tau[I/i]\ \sigma\iota]\!]$ and we are done.

28. T-CI:

$$\frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (c \Rightarrow \tau)} \text{ T-CI}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l, T, \Lambda.e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall v, t < T\ .\Lambda.e\ \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

This means given some $v, t < T$ s.t $\Lambda.e\ \delta\gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e\ \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$(p_l, T, \Lambda.e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that

$. \models c\ \iota \implies (p_l, T, e\ \delta\gamma) \in [\![\tau\sigma\iota]\!]_{\mathcal{E}}$

This means given that $. \models c\ \iota$ it suffices to prove that

$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

<u>IH</u> $(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

We get the desired directly from IH

29. T-CE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e[] : \tau} \text{T-CE}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma \sigma \iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma \iota]\!]_{\mathcal{E}}$ and $\models \Delta\ \iota$

To prove: $(p_l, T, e\ []\ \delta\gamma) \in [\![(\tau)\ \sigma \iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall v_f, t < T\ .(e\ [])\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![(\tau)\ \sigma \iota]\!]$

This means given some $v_f, t < T$ s.t $(e\ [])\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l, T - t, v_f) \in [\![(\tau)\ \sigma \iota]\!]$ (F-Tap0)

<u>IH</u>

$(p_l, T, e\ \delta\gamma) \in [\![(c \Rightarrow \tau)\ \sigma \iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall v', t' < T\ .e\ \delta\gamma \Downarrow v' \implies (p_l + p_m, v') \in [\![(c \Rightarrow \tau)\ \sigma \iota]\!]$

Since we know that $(e\ [])\ \delta\gamma \Downarrow_t v_f$ therefore from E-CE we know that $\exists t' < t.e\delta\gamma \Downarrow_{t'} \Lambda.e'$, an since $t' < t < T$ therefore we have

$(p_l, T - t', \Lambda.e') \in [\![(c \Rightarrow \tau)\ \sigma \iota]\!]$

Therefore from Definition 66 we have

$. \models c\ \iota \implies (p_l, T - t', e'\ \delta\gamma) \in [\![\tau\ \sigma \iota]\!]_{\mathcal{E}}$

Since we are given $\Theta; \Delta \models c$ and $. \models \Delta\ \iota$ therefore we know that $. \models c\ \iota$. Hence we get

$(p_l, T - t', e'\ \delta\gamma) \in [\![\tau\ \sigma \iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall v'_f, t'' < T - t'.(e')\ \delta\gamma \Downarrow_{t''} v'_f \implies (p_l, T - t' - t'', v'_f) \in [\![(\tau)\ \sigma \iota]\!]$ (F-CE1)

Since from E-CE we know that $e'\delta\gamma \Downarrow_t v_f$ therefore we know that $\exists t''.e'\ \delta\gamma \Downarrow_{t''} v_f$ s.t $t = t' + t'' + 1$

Therefore instantiating (F-CE1) with the given v_f and t'' we get

$(p_l, T - t' - t'', v_f) \in [\![(\tau)\ \sigma \iota]\!]$

Since $t = t' + t'' + 1$ therefore from Lemma 69 we get the desired.

30. T-CAndI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{T-CAndI}$$

Given: $(p_l, T\ \gamma) \in [\![\Gamma\ \sigma \iota]\!]_{\mathcal{E}}$, $(0, T\ \delta) \in [\![\Omega\ \sigma \iota]\!]_{\mathcal{E}}$

To prove: $(p_l, e\ \delta\gamma) \in [\![c \& \tau\ \sigma \iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall v_f, t < T . e\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f\ \delta\gamma) \in [\![c \& \tau\ \sigma\iota]\!]$

This means given some $v_f, t < T$ s.t $e\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![c \& \tau\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$. \models c\iota \wedge (p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we are given that $. \models \Delta\iota$ and $\Theta; \Delta \models c$ therefore it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-CAI0)

IH: $(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t' < T . e\ \delta\gamma \Downarrow_{t'} v_f \implies (p_l, T - t', v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we are given that $e\ \delta\gamma \Downarrow_t v_f$ therefore we get
$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-CAI1)

We get the desired from (F-CAI1)

31. T-CAndE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (c \& \tau) \qquad \Psi; \Theta; \Delta, c; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{clet}\, x = e \,\mathsf{in}\, e' : \tau'} \text{ T-CAndE}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\ \sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, (\mathsf{clet}\, x = e \,\mathsf{in}\, e')\ \delta\gamma) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall v_f, t < T . (\mathsf{clet}\, x = e \,\mathsf{in}\, e')\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

This means given soem $v_f, t < T$ s.t $(\mathsf{clet}\, x = e \,\mathsf{in}\, e')\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that
$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$ (F-CAE0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2} . p_{l1} + p_{l2} = p_l$ s.t
$(p_{l1}, T, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

IH1
$(p_{l1}, T, e\ \delta\gamma) \in [\![c \& \tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_1 < T . e\ \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1 v_1) \in [\![c \& \tau\ \sigma\iota]\!]_{\mathcal{E}}$

Since we know that $(\mathsf{clet}\, x = e \,\mathsf{in}\, e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists v_1, t_1 < t . e\ \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have
$(p_{l1}, T - t_1, v_1) \in [\![c \& \tau\ \sigma\iota]\!]$

Therefore from Definition 66 we have

$. \models c\iota \wedge (p_{l1}, T - t_1, v_1) \in [\![\tau\ \sigma\iota]\!]$ (F-CAE1)

<u>IH2</u>

$(p_{l2} + p_{l1}, T, e'\ \delta\gamma') \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v_1\}$

This means from Definition 66 we have

$\forall t_2 < T . e'\ \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{l1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since we know that $(\mathsf{clet}\, x = e \text{ in } e')\ \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_2 < t. e'\ \delta'\gamma \Downarrow_{t_2} v_f$.

Therefore we have

$(p_{l2} + p_{l1}, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

Since $p_l = p_{l1} + p_{l2}$ therefore we get

$(p_l, T - t_2, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And finally from From Lemma 69 we get

$(p_l, T - t, v_f) \in [\![\tau'\ \sigma\iota]\!]$

And we are done.

32. T-fix:

$$\frac{\Psi; \Theta; \Delta; \Omega, x : \tau; . \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; . \vdash \mathsf{fix} x.e : \tau} \text{ T-fix}$$

Given: $(0, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(0, T, (\mathsf{fix} x.e)\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ (F-FX0)

We induct on T

<u>Base case, $T = 1$</u>:

It suffices to prove that $(0, 1, (\mathsf{fix} x.e)\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]$

This means from Definition 66 it suffices to prove

$\forall t < 1. (\mathsf{fix} x.e)\ \delta\gamma \Downarrow_t v \implies (0, 1 - t, v) \in [\![\tau]\!]$

This further means that given $t < 1$ s.t $(\mathsf{fix} x.e)\ \delta\gamma \Downarrow_t v$ it suffices to prove that

$(0, 1 - t, v) \in [\![\tau]\!]$

Since from E-fix we know that minimum value of t can be 1 therefore $t < 1$ is not possible and the goal holds vacuously.

<u>Inductive case</u>:

IH: $(0, T - 1, (\mathsf{fix} x.e)\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

Therefore from Definition 67 we have

$(0, T-1, \delta') \in [\![\Omega, x : \tau\ \sigma\iota]\!]_{\mathcal{E}}$ where $\delta' = \delta \cup \{x \mapsto \mathsf{fix} x.e\ \delta\}$

Applying Definition 66 on (F-FX0) it suffices to prove that

$\forall t < T\ .(\mathsf{fix} x.e)\ \delta\gamma \Downarrow_t v_f \implies (0, T-t, v_f) \in [\![\tau\ \sigma\iota]\!]$

This means given some $t < T$ s.t $\mathsf{fix} x.e\ \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(0, T-t, v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-FX0.0)

Now from IH of outer induction we have

$(0, T-1, e\ \delta'\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t' < T-1.e\ \delta'\gamma \Downarrow_{t'} v_f \implies (0, T-1-t', v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $\mathsf{fix} x.e\ \delta\gamma \Downarrow_t v_f$ therefore from E-fix we know that $\exists t' = t-1$ s.t $e\ \delta'\gamma \Downarrow_{t'} v_f$

Since $t < T$ therefore $t' = t-1 < T-1$ hence we have

$(0, T-t, v_f) \in [\![\tau\ \sigma\iota]\!]$

Therefore we are done

33. T-ret:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{ret}\, e : \mathbb{M}\, 0\, \tau} \text{ T-ret}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, \mathsf{ret}\, e\ \delta\gamma) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f.(\mathsf{ret}\, e)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T-t, v_f) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]$

It means we are given some $t < T, v_f$ s.t $(\mathsf{ret}\, e)\ \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\mathsf{ret}\, e)\ \delta\gamma$.

Therefore it suffices to prove that

$(p_l, T, (\mathsf{ret}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, 0\, \tau\ \sigma\iota]\!]$

From Definition 66 it further suffices to prove that

$\forall t' < T\ .(\mathsf{ret}\, e)\ \delta\gamma \Downarrow_{t'}^{n'} v_f \implies \exists p'.n' + p' \leqslant p_l \wedge (p', T-t', v_f) \in [\![\tau\ \sigma\iota]\!]$

This means given some $t' < T$ s.t $(\mathsf{ret}\, e)\ \delta\gamma \Downarrow_{t'}^{n'} v_f$ it suffices to prove that

$\exists p'.n' + p' \leqslant p_l \wedge (p', T-t', v_f) \in [\![\tau\ \sigma\iota]\!]$

From (E-ret) we know that $n' = 0$ therefore we choose p' as p_l and it suffices to prove that

$(p_l, T-t', v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-R0)

IH

$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T\ .(e)\ \delta\gamma \Downarrow_t v_f \implies (p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{ret}\, e)\ \delta\gamma \Downarrow^0_{t'} v_f$ therefore from (E-ret) we know that $\exists t_1.e\ \delta\gamma \Downarrow_{t_1} v_f$

Since $t_1 < t < T$ therefore we have

$(p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

And finally from Lemma 69 we get

$(p_l, T - t, v_f) \in [\![\tau\ \sigma\iota]\!]$

and we are done.

34. T-bind:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \mathbb{M}\, n_1\, \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}\, n_2\, \tau_2 \qquad \Theta \vdash n_1 : \mathbb{R}^+ \qquad \Theta \vdash n_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2 : \mathbb{M}(n_1 + n_2)\, \tau_2} \text{ T-bind}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, \mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2\ \delta\gamma) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v.(\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2)\ \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

This means given some $t < T, v$ s.t $(\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2)\ \delta\gamma \Downarrow_t v$. From E-val we know that $t = 0$ and $v = (\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2\ \delta\gamma)$

Therefore it suffices to prove that

$(p_l, T, (\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2\ \delta\gamma)) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 66 it suffices to prove that

$\forall t' < T, v_f.(\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2\ \delta\gamma) \Downarrow^{s'}_{t'} v_f \implies \exists p'.s' + p' \leqslant p_l + n \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$

This means given some $t' < T, v_f$ s.t $(\mathsf{bind}\, x = e_1 \,\mathsf{in}\, e_2\ \delta\gamma) \Downarrow^{s'}_{t'} v_f$ and we need to prove that

$\exists p'.s' + p' \leqslant p_l + n \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-B0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

IH1

$(p_{l1}, T, e_1\ \delta\gamma) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it means we have

$\forall t_1 < T\ .(e_1)\ \delta\gamma \Downarrow_{t_1} v_{m1} \implies (p_{l1}, T - t_1, v_{m1}) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow^{s'}_{t'} v_f$ therefore from E-bind we know that $\exists t_1 < t', v_{m1}.(e_1)\ \delta\gamma \Downarrow_{t_1} v_{m1}$.

Since $t_1 < t' < T$, therefore we have

$(p_{l1}, T - t_1, v_{m1}) \in [\![\mathbb{M}(n_1)\, \tau_1\ \sigma\iota]\!]$ (F-B1)

This means from Definition 66 we are given that

$\forall t'_1 < T - t_1.v_{m1} \Downarrow^{s_1} v_1 \implies \exists p'_1.s_1 + p'_1 \leqslant p_{l1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in [\![\tau_1\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow_{t'} v_f$ therefore from E-bind we know that $\exists t'_1 < t - t_1.(e_1)\ \delta\gamma \Downarrow^{s_1}_{t'_1} v_1$.

Since $t'_1 < t - t_1 < T - t_1$ therefore means we have

$\exists p'_1.s_1 + p'_1 \leqslant p_{l1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in [\![\tau_1\ \sigma\iota]\!]$ (F-B1)

<u>IH2</u>

$(p_{l2} + p'_1, T - t_1 - t'_1, e_2\ \delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it means we have

$\forall t_2 < T - t_1 - t'_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2} \implies (p_{l2} + p'_1, T - t_1 - t'_1 - t_2, v_{m2}) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow^{s'}_{t'} v_f$ therefore from E-bind we know that $\exists t_2 < t' - t_1 - t'_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2}$.

Since $t_2 < t' - t_1 - t'_1 < T - t_1 - t'_1$ therefore we have

$(p_{l2} + p'_1, T - t_1 - t'_1 - t_2, v_{m2}) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 66 we are given that

$\forall t'_2 < T - t_1 - t'_1 - t_2.v_{m2} \Downarrow^{s_2}_{t'_2} v_2 \implies \exists p'_2.s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$

Since we know that $(\mathsf{bind}\, x = e_1 \;\mathsf{in}\; e_2)\ \delta\gamma \Downarrow^{s'}_{t'} v_f$ therefore from E-bind we know that $\exists t'_2 < t' - t_1 - t'_1 - t_2, s_2, v_2.v_{m2} \Downarrow^{s_2}_{t'_2} v_2$.

This means we have

$\exists p'_2.s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-B2)

In order to prove (F-B0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leqslant p_l + n$:
Since from (F-B2) we know that
$s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_2$

Adding s_1 on both sides we get
$s_1 + s_2 + p'_2 \leqslant p_{l2} + s_1 + p'_1 + n_2$

Since from (F-B1) we know that
$s_1 + p'_1 \leqslant p_{l1} + n_1$

therefore we also have
$s_1 + s_2 + p'_2 \leqslant p_{l2} + p_{l1} + n_1 + n_2$

And finally since we know that $n = n_1 + n_2$, $s' = s_1 + s_2$ and $p_l = p_{l1} + p_{l2}$ therefore we get the desired

(b) $(p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_f) \in [\![\tau_2\ \sigma\iota]\!]$:
From E-bind we know that $v_f = v_2$ therefore we get the desired from (F-B2)

35. T-tick:

$$\frac{\Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^n : \mathbb{M}\, n\, \mathbf{1}} \text{ T-tick}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, \uparrow^n\ \delta\gamma) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that
$\forall t < T, v.(\uparrow^n)\ \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]$

This means we are given some $t < T, v$ s.t $(\uparrow^n)\ \delta\gamma \Downarrow_t v$. From E-val we know that $t = 0$ and $v = (\uparrow^n)\ \delta\gamma$
Therefore it suffices to prove that
$(p_l, T, (\uparrow^n)\ \delta\gamma) \in [\![\mathbb{M}\, n\, \mathbf{1}\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that
$\forall t' < T\ .(\uparrow^n)\ \delta\gamma \Downarrow_{t'}^{n'} () \implies \exists p'.n' + p' \leqslant p_l + n \wedge (p', T - t', ()) \in [\![\mathbf{1}]\!]$

This means given some $t' < T$ s.t $(\uparrow^n)\ \delta\gamma \Downarrow_{t'}^{n'} ()$ it suffices to prove that
$\exists p'.n' + p' \leqslant p_l + n \wedge (p', T - t', ()) \in [\![\mathbf{1}]\!]$

From (E-tick) we know that $n' = n$ therefore we choose p' as p_l and it suffices to prove that
$(p_l, T - t', ()) \in [\![\mathbf{1}]\!]$

We get this directly from Definition 66

36. T-release:

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : [n_1]\,\tau_1 \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(n_1 + n_2)\,\tau_2 \qquad \Theta \vdash n_1 : \mathbb{R}^+ \qquad \Theta \vdash n_2 : \mathbb{R}^+\end{array}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \mathsf{release}\, x = e_1\ \mathsf{in}\ e_2 : \mathbb{M}\, n_2\, \tau_2} \text{ T-release}$$

Given: $(p_l, T, \gamma) \in [\![(\Gamma_1 \oplus \Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![(\Omega)\ \sigma\iota]\!]_{\mathcal{E}}$
To prove: $(p_l, T, \mathsf{release}\, x = e_1\ \mathsf{in}\ e_2\ \delta\gamma) \in [\![\mathbb{M}(n_2)\,\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v.(\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

This means given some $t < T, v$ s.t $(\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma \Downarrow (\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma$. From E-val we know that $t = 0$ and $v = (\textsf{release}\, x = e_1 \textsf{ in } e_2\ \delta\gamma)$

Therefore it suffices to prove that

$(p_l, T, (\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma) \in [\![\mathbb{M}(n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 66 it suffices to prove that

$\forall t' < T, v_f.(\textsf{release}\, x = e_1 \textsf{ in } e_2\ \delta\gamma) \Downarrow_{t'}^{s'} v_f \implies \exists p'.s' + p' \leqslant p_l + n_2 \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$

This means given some $t' < T, v_f$ s.t $(\textsf{release}\, x = e_1 \textsf{ in } e_2\ \delta\gamma) \Downarrow_{t'}^{s'} v_f$ and we need to prove that

$\exists p'.s' + p' \leqslant p_l + n_2 \wedge (p', T - t', v_f) \in [\![\tau_2\ \sigma\iota]\!]$ (F-R0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}.p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in [\![(\Gamma_1)\sigma\iota]\!]_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in [\![(\Gamma_2)\sigma\iota]\!]_{\mathcal{E}}$

<u>IH1</u>

$(p_{l1}, T, e_1\ \delta\gamma) \in [\![[n_1]\, \tau_1\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it means we have

$\forall t_1 < T\ .(e_1)\ \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in [\![[n_1]\, \tau_1\ \sigma\iota]\!]$

Since we know that $(\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-rel we know that $\exists t_1 < t'.(e_1)\ \delta\gamma \Downarrow_{t_1} v_1$.

Since $t_1 < t' < T$, therefore we have

$(p_{l1}, T - t_1, v_1) \in [\![[n_1]\, \tau_1\ \sigma\iota]\!]$

This means from Definition 66 we have

$\exists p'_1.p'_1 + n_1 \leqslant p_{l1} \wedge (p'_1, T - t_1, v_1) \in [\![\tau_1]\!]$ (F-R1)

<u>IH2</u>

$(p_{l2} + p'_1, T - t_1, e_2\ \delta\gamma \cup \{x \mapsto v_1\}) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it means we have

$\forall t_2 < T - t_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2} \cup \{x \mapsto v_1\} \implies (p_{l2} + p'_1, T - t_1 - t_2, v_{m2}) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

Since we know that $(\textsf{release}\, x = e_1 \textsf{ in } e_2)\ \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-rel we know that $\exists t_2 < t - t_1.(e_2)\ \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2}$. This means we have

$(p_{l2} + p'_1, T - t_1 - t_2, v_{m2}) \in [\![\mathbb{M}(n_1 + n_2)\, \tau_2\ \sigma\iota]\!]$

This means from Definition 66 we are given that

$\forall t'_2 < T - t_1 - t_2.v_{m2} \Downarrow_{t'_2}^{s_2} v_2 \implies \exists p'_2.s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$

Since we know that $(\mathsf{release}\, x = e_1 \,\mathsf{in}\, e_2)\ \delta\gamma \Downarrow^{s'}_{t'} v_f$ therefore from E-rel we know that $\exists t'_2 . v_{m2} \Downarrow^{s_2}_{t'_2} v_2$ s.t. $t'_2 = t' - t_1 - t_2 - 1$

Since $t'_2 = t' - t_1 - t_2 < T - t_1 - t_2$, therefore we have

$\exists p'_2 . s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-R2)

In order to prove (F-R0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leqslant p_l + n_2$:
Since from (F-R2) we know that
$s_2 + p'_2 \leqslant p_{l2} + p'_1 + n_1 + n_2$

Since from (F-R1) we know that
$p'_1 + n_1 \leqslant p_{l1}$

therefore we also have
$s_2 + p'_2 \leqslant p_{l2} + p_{l1} + p_{m1} + n_2$

And finally since we know that $s' = s_2$, $p_l = p_{l1} + p_{l2}$ and $0 = p_{m1}$ therefore we get the desired

(b) $(p'_2, T - t_1 - t_2 - t'_2, v_f) \in [\![\tau_2\ \sigma\iota]\!]$:
From E-rel we know that $v_f = v_2$ therefore we get the desired from (F-R2)

37. T-store:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \qquad \Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathsf{store}\, e : \mathbb{M}\, n\, ([n]\, \tau)} \text{ T-store}$$

Given: $(p_l, T, \gamma) \in [\![\Gamma\sigma\iota]\!]_{\mathcal{E}}$, $(0, T, \delta) \in [\![\Omega\ \sigma\iota]\!]_{\mathcal{E}}$

To prove: $(p_l, T, \mathsf{store}\, e\ \delta\gamma) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v . (\mathsf{store}\, e)\ \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]$

This means we are given some $t < T, v$ s.t $(\mathsf{store}\, e)\ \delta\gamma \Downarrow_t v$. From E-val we know that $t = 0$ and $v = (\mathsf{store}\, e)\ \delta\gamma$

Therefore it suffices to prove that

$(p_l, T, (\mathsf{store}\, e)\ \delta\gamma) \in [\![\mathbb{M}\, n\, ([n]\, \tau)\ \sigma\iota]\!]$

From Definition 66 it suffices to prove that

$\forall t' < T, v_f, n' . (\mathsf{store}\, e)\ \delta\gamma \Downarrow^{n'}_{t'} v_f \implies \exists p' . n' + p' \leqslant p_l \wedge (p', T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]$

This means given some $t' < T, v_f$ s.t $(\mathsf{store}\, e)\ \delta\gamma \Downarrow^{n'}_{t'} v_f$ it suffices to prove that

$\exists p' . n' + p' \leqslant p_l \wedge (p', T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]$

From (E-store) we know that $n' = 0$ therefore we choose p' as $p_l + n$ and it suffices to prove that

$(p_l + n, T - t', v_f) \in [\![[n]\, \tau\ \sigma\iota]\!]$

This further means that from Definition 66 we have
$\exists p''.p'' + n \leqslant p_l + n \wedge (p'', T - t', v_f) \in [\![\tau\ \sigma\iota]\!]$

We choose p'' as p_l and it suffices to prove that
$(p_l, T - t', v_f) \in [\![\tau\ \sigma\iota]\!]$ (F-S0)

IH
$(p_l, T, e\ \delta\gamma) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we have
$\forall t_1 < T .(e)\ \delta\gamma \Downarrow_{t_1} v_f \implies (p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

Since we know that $(\mathsf{store}\, e)\ \delta\gamma \Downarrow^0_{t'} v_f$ therefore from (E-store) we know that $\exists t_1 < t'.e\ \delta\gamma \Downarrow_{t_1} v_f$

Since $t_1 < t' < T$ therefore we have
$(p_l, T - t_1, v_f) \in [\![\tau\ \sigma\iota]\!]$

and finally from Lemma 69 we have
$(p_l, T - t', v_f) \in [\![\tau\ \sigma\iota]\!]$

$\square$

Lemma 72 (Value subtyping lemma)**.** $\forall \Psi, \Theta, \tau \in Type, \tau'.$
$\Psi; \Theta; \Delta \vdash \tau <: \tau' \wedge . \models \Delta\iota \implies [\![\tau\ \sigma\iota]\!] \subseteq [\![\tau'\ \sigma\iota]\!]$

Proof. Proof by induction on the $\Psi; \Theta; \Delta \vdash \tau <: \tau'$ relation

1. sub-refl:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{ sub-refl}$$

To prove: $\forall (p, T, v) \in [\![\tau\ \sigma\iota]\!] \implies (p, T, v) \in [\![\tau\ \sigma\iota]\!]$

Trivial

2. sub-arrow:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1 \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau_1' \multimap \tau_2'} \text{ sub-arrow}$$

To prove: $\forall (p, T, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!] \implies (p, T, \lambda x.e) \in [\![(\tau_1' \multimap \tau_2')\ \sigma\iota]\!]$

This means given some $(p, T, \lambda x.e) \in [\![(\tau_1 \multimap \tau_2)\ \sigma\iota]\!]$ we need to prove
$(p, T, \lambda x.e) \in [\![(\tau_1' \multimap \tau_2')\ \sigma\iota]\!]$

From Definition 66 we are given that
$\forall\ T' < T, p', e'.(p', T', e') \in [\![\tau_1\ \sigma\iota]\!]_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SL0)

Also from Definition 66 it suffices to prove that

$\forall\, T''{<}T, p'', e''.(p'', T'', e'') \in [\![\tau_1'\ \sigma\iota]\!]_{\mathcal{E}} \implies (p + p'', T'', e[e''/x]) \in [\![\tau_2'\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $T''{<}T, p'', e''$ s.t $(p'', T'', e'') \in [\![\tau_1'\ \sigma\iota]\!]$ we need to prove
$(p + p', T'', e[e''/x]) \in [\![\tau_2'\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SL1)

<u>IH1</u>: $[\![\tau_1'\ \sigma\iota]\!] \subseteq [\![\tau_1\ \sigma\iota]\!]$

Since we have $(p'', T'', e'') \in [\![\tau_1'\ \sigma\iota]\!]$ therefore from IH1 we also have $(p'', T'', e'') \in [\![\tau_1\ \sigma\iota]\!]$

Therefore instantiating (F-SL0) with p', T'', e'' we get
$(p + p'', T'', e[e''/x]) \in [\![\tau_2\ \sigma\iota]\!]_{\mathcal{E}}$

And finally from Lemma 73 we get the desired

3. sub-tensor:

$$\dfrac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau_1' \otimes \tau_2'}\ \text{sub-tensor}$$

To prove: $\forall (p, T, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1' \otimes \tau_2')\ \sigma\iota]\!]$

This means given $(p, T, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1 \otimes \tau_2)\ \sigma\iota]\!]$
It suffices prove that
$(p, T, \langle\!\langle v_1, v_2\rangle\!\rangle) \in [\![(\tau_1' \otimes \tau_2')\ \sigma\iota]\!]$

This means from Definition 66 we are given that
$\exists p_1, p_2. p_1 + p_2 \leqslant p \wedge (p_1, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p_2, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$

Also from Definition 66 it suffices to prove that
$\exists p_1', p_2'. p_1' + p_2' \leqslant p \wedge (p_1', T, v_1) \in [\![\tau_1'\ \sigma\iota]\!] \wedge (p_2', T, v_2) \in [\![\tau_2'\ \sigma\iota]\!]$

<u>IH1</u> $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_1')\ \sigma\iota]\!]$
<u>IH2</u> $[\![(\tau_2)\ \sigma\iota]\!] \subseteq [\![(\tau_2')\ \sigma\iota]\!]$

Choosing p_1 for p_1' and p_2 for p_2' we get the desired from IH1 and IH2

4. sub-with:

$$\dfrac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \,\&\, \tau_2 <: \tau_1' \,\&\, \tau_2'}\ \text{sub-with}$$

To prove: $\forall (p, T, \langle v_1, v_2\rangle) \in [\![(\tau_1 \,\&\, \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle v_1, v_2\rangle) \in [\![(\tau_1' \,\&\, \tau_2')\ \sigma\iota]\!]$

This means given $(p, T, \langle v_1, v_2\rangle) \in [\![(\tau_1 \,\&\, \tau_2)\ \sigma\iota]\!]$
It suffices prove that
$(p, T, \langle v_1, v_2\rangle) \in [\![(\tau_1' \,\&\, \tau_2')\ \sigma\iota]\!]$

This means from Definition 66 we are given that

$(p, T, v_1) \in [\![\tau_1\ \sigma\iota]\!] \wedge (p, T, v_2) \in [\![\tau_2\ \sigma\iota]\!]$ (F-SW0)

Also from Definition 66 it suffices to prove that
$(p, T, v_1) \in [\![\tau_1'\ \sigma\iota]\!] \wedge (p, T, v_2) \in [\![\tau_2'\ \sigma\iota]\!]$

<u>IH1</u> $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_1')\ \sigma\iota]\!]$
<u>IH2</u> $[\![(\tau_2)\ \sigma\iota]\!] \subseteq [\![(\tau_2')\ \sigma\iota]\!]$

We get the desired from (F-SW0), IH1 and IH2

5. sub-sum:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \qquad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau_1' \oplus \tau_2'} \text{ sub-sum}$$

To prove: $\forall (p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!] \implies (p, T, \langle\!\langle v_1, v_2 \rangle\!\rangle) \in [\![(\tau_1' \oplus \tau_2')\ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![(\tau_1 \oplus \tau_2)\ \sigma\iota]\!]$
It suffices prove that
$(p, T, v) \in [\![(\tau_1' \oplus \tau_2')\ \sigma\iota]\!]$

This means from Definition 66 two cases arise

(a) $v = \mathsf{inl}(v')$:
This means from Definition 66 we have $(p, T, v') \in [\![\tau_1\ \sigma\iota]\!]$ (F-SS0)

Also from Definition 66 it suffices to prove that
$(p, T, v') \in [\![\tau_1'\ \sigma\iota]\!]$

<u>IH</u> $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_1')\ \sigma\iota]\!]$

We get the desired from (F-SS0), IH

(b) $v = \mathsf{inr}(v')$:
Symmetric reasoning as in the inl case

6. sub-list:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n\ \tau <: L^n\ \tau'} \text{ sub-list}$$

To prove: $\forall (p, T, v) \in [\![L^n\ \tau\ \sigma\iota]\!].(p, T, v) \in [\![L^n\ \tau'\ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![L^n\ \tau\ \sigma\iota]\!]$ and we need to prove
$(p, T, v) \in [\![L^n\ \tau'\ \sigma\iota]\!]$

We induct on $(p, T, v) \in [\![L^n\ \tau\ \sigma\iota]\!]$

(a) $(p, T, nil) \in [\![L^0\ \tau\ \sigma\iota]\!]$:
We need to prove $(p, T, nil) \in [\![L^0\ \tau'\ \sigma\iota]\!]$
We get this directly from Definition 66

(b) $(p, T, v' :: l') \in [\![L^{m+1}\ \tau\ \sigma\iota]\!]$:
In this case we are given $(p, T, v' :: l') \in [\![L^{m+1}\ \tau\ \sigma\iota]\!]$
and we need to prove $(p, T, v' :: l') \in [\![L^{m+1}\ \tau'\ \sigma\iota]\!]$

This means from Definition 66 are given
$\exists p_1, p_2.p_1 + p_2 \leqslant p \wedge (p_1, T, v') \in [\![\tau\ \sigma\iota]\!] \wedge (p_2, T, l') \in [\![L^m\tau\ \sigma\iota]\!]$ (Sub-List0)

Similarly from Definition 66 we need to prove that
$\exists p'_1, p'_2.p'_1 + p'_2 \leqslant p \wedge (p'_1, T, v') \in [\![\tau'\ \sigma\iota]\!] \wedge (p_2, T, l') \in [\![L^m\tau'\ \sigma\iota]\!]$

We choose p'_1 as p_1 and p'_2 as p_2 and we get the desired from (Sub-List0) IH of outer induction and IH of innner induction

7. sub-exist:

$$\frac{\Psi; \Theta, s; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s.\tau <: \exists s.\tau'} \text{ sub-exist}$$

To prove: $\forall (p, T, v) \in [\![\exists s.\tau\ \sigma\iota]\!].(p, T, v) \in [\![\exists s.\tau'\ \sigma\iota]\!]$

This means given some $(p, T, v) \in [\![\exists s.\tau\ \sigma\iota]\!]$ we need to prove
$(p, T, v) \in [\![\exists s.\tau'\ \sigma\iota]\!]$

From Definition 66 we are given that
$\exists s'.(p, T, v) \in [\![\tau\sigma\iota[s'/s]]\!]$ (F-exist0)

<u>IH</u>: $[\![(\tau)\ \sigma\iota \cup \{s \mapsto s'\}]\!] \subseteq [\![(\tau')\ \sigma\iota \cup \{s \mapsto s'\}]\!]$

Also from Definition 66 it suffices to prove that
$\exists s''.(p, T, v) \in [\![\tau'\sigma\iota[s''/s]]\!]$

We choose s'' as s' and we get the desired from IH

8. sub-potential:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Psi; \Theta; \Delta \vdash n' \leqslant n}{\Psi; \Theta; \Delta \vdash [n]\,\tau <: [n']\,\tau'} \text{ sub-potential}$$

To prove: $\forall (p, T, v) \in [\![[n]\,\tau\ \sigma\iota]\!].(p, T, v) \in [\![[n']\,\tau'\ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![[n]\,\tau\ \sigma\iota]\!]$ and we need to prove
$(p, T, v) \in [\![[n']\,\tau'\ \sigma\iota]\!]$

This means from Definition 66 we are given
$\exists p'.p' + n \leqslant p \wedge (p', T, v) \in [\![\tau\ \sigma\iota]\!]$ (F-SP0)

And we need to prove
$\exists p''.p'' + n' \leqslant p \wedge (p'', T, v) \in [\![\tau' \ \sigma\iota]\!]$ (F-SP1)

In order to prove (F-SP1) we choose p'' as p'

Since from (F-SP0) we know that $p' + n \leqslant p$ and we are given that $n' \leqslant n$ therefore we also have $p' + n' \leqslant p$

<u>IH</u> $[\![\tau \ \sigma\iota]\!] \subseteq [\![\tau' \ \sigma\iota]\!]$

We get the desired directly from IH

9. sub-monad:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \qquad \Psi; \Theta; \Delta \vdash n \leqslant n'}{\Psi; \Theta; \Delta \vdash \mathbb{M}\, n\, \tau <: \mathbb{M}\, n'\, \tau'} \text{ sub-monad}$$

To prove: $\forall (p, T, v) \in [\![\mathbb{M}\, n\, \tau \ \sigma\iota]\!].(p, T, v) \in [\![\mathbb{M}\, n'\, \tau' \ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![\mathbb{M}\, n\, \tau \ \sigma\iota]\!]$ and we need to prove
$(p, T, v) \in [\![\mathbb{M}\, n'\, \tau' \ \sigma\iota]\!]$

This means from Definition 66 we are given
$\forall t' < T, n_1, v'.v \Downarrow_{t'}^{n_1} v' \implies \exists p'.n_1 + p' \leqslant p + n \wedge (p', T - t', v') \in [\![\tau \ \sigma\iota]\!]$ (F-SM0)

Again from Definition 66 we need to prove that
$\forall t'' < T, n_2, v''.v \Downarrow_{t''}^{n_2} v'' \implies \exists p''.n_2 + p'' \leqslant p + n' \wedge (p'', T - t'', v'') \in [\![\tau' \ \sigma\iota]\!]$

This means given some $t'' < T, v'', n_2$ s.t $v \Downarrow_{t''}^{n_2} v'$ it suffices to prove that
$\exists p''.n_2 + p'' \leqslant p + n' \wedge (p'', T - t'', v'') \in [\![\tau' \ \sigma\iota]\!]$ (F-SM1)

Instantiating (F-SM0) with t'', n_2, v'' Since $v \Downarrow_{t''}^{n_2} v''$ therefore from (F-SM0) we know that
$\exists p'.n_2 + p' \leqslant p + n \wedge (p', T - t'', v'') \in [\![\tau \ \sigma\iota]\!]$ (F-SM2)

<u>IH</u> $[\![\tau \ \sigma\iota]\!] \subseteq [\![\tau' \ \sigma\iota]\!]$

In order to prove (F-SM1) we choose p'' as p' and we need to prove

(a) $n_2 + p'' \leqslant p + n'$:
Since we are given that $n \leqslant n'$ therefore we get the desired from (F-SM2)

(b) $(p', v') \in [\![\tau' \ \sigma\iota]\!]$
We get this directly from IH and (F-SM2)

10. sub-Exp:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash !\tau <: !\tau'} \text{ sub-Exp}$$

To prove: $\forall (p, T, v) \in [\![!\tau\ \sigma\iota]\!].(p, T, v) \in [\![!\tau'\ \sigma\iota]\!]$

This means given $(p, T, !e) \in [\![!\tau\ \sigma\iota]\!]$ and we need to prove
$(p, T, !e) \in [\![!\tau'\ \sigma\iota]\!]$

This means from Definition 66 we are given
$(0, T, e) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SE0)

Again from Definition 66 we need to prove that
$(0, T, e) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$ (F-SE1)

<u>IH</u> $[\![\tau\ \sigma\iota]\!] \subseteq [\![\tau'\ \sigma\iota]\!]$
Therefore from (F-SE0) and IH we get $(0, T, e) \in [\![\tau'\ \sigma\iota]\!]$ and we are done.

11. sub-typePoly:

$$\frac{\Psi, \alpha; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha.\tau_1 <: \forall i.\tau_2} \text{ sub-typePoly}$$

To prove: $\forall (p, T, \Lambda.e) \in [\![(\forall i.\tau_1)\ \sigma\iota]\!].(p, T, \Lambda.e) \in [\![(\forall i.\tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \Lambda.e) \in [\![(\forall \alpha.\tau_1)\ \sigma\iota]\!]$ we need to prove
$(p, T, \Lambda.e) \in [\![(\forall \alpha.\tau_2)\ \sigma\iota]\!]$

From Definition 66 we are given that
$\forall \tau, T'{<}T\ .(p, T', e) \in [\![\tau_1[\tau/\alpha]]\!]_{\mathcal{E}}$ (F-STP0)

Also from Definition 66 it suffices to prove that
$\forall \tau', T''{<}T\ .(p, T'', e) \in [\![\tau_2[\tau'/\alpha]]\!]_{\mathcal{E}}$

This means given some $\tau', T''{<}T$ and we need to prove
$(p, T'', e) \in [\![\tau_2[\tau'/\alpha]]\!]_{\mathcal{E}}$ (F-STP1)

<u>IH</u>: $[\![(\tau_1)\ \sigma\iota \cup \{\alpha \mapsto \tau'\}]\!] \subseteq [\![(\tau_2)\ \sigma\iota \cup \{\alpha \mapsto \tau'\}]\!]$

Instantiating (F-STP0) with τ', T'' we get
$(p, T'', e) \in [\![\tau_1[\tau'/\alpha]]\!]_{\mathcal{E}}$

and finally from IH we get the desired.

12. sub-indexPoly:

$$\frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i.\tau_1 <: \forall i.\tau_2} \text{ sub-indexPoly}$$

To prove: $\forall (p, T, \Lambda i.e) \in [\![(\forall i.\tau_1)\ \sigma\iota]\!].(p, T, \Lambda i.e) \in [\![(\forall i.\tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \Lambda i.e) \in [\![(\forall i.\tau_1)\ \sigma\iota]\!]$ we need to prove

$(p, T, \Lambda i.e) \in [\![(\forall i.\tau_2)\ \sigma\iota]\!]$

From Definition 66 we are given that
$\forall I, T'{<}T\ .(p, T', e) \in [\![\tau_1[I/i]]\!]_{\mathcal{E}}$ (F-SIP0)

Also from Definition 66 it suffices to prove that
$\forall I', T''{<}T\ .(p, T'', e) \in [\![\tau_2[I'/i]]\!]_{\mathcal{E}}$

This means given some $I', T''{<}T$ and we need to prove
$(p, T'', e) \in [\![\tau_2[I'/i]]\!]_{\mathcal{E}}$ (F-SIP1)

IH: $[\![(\tau_1)\ \sigma\iota \cup \{i \mapsto I'\}]\!] \subseteq [\![(\tau_2)\ \sigma\iota \cup \{i \mapsto I'\}]\!]$

Instantiating (F-SIP0) with I', T'' we get
$(p, T'', e) \in [\![\tau_1[I'/i]]\!]_{\mathcal{E}}$

and finally from IH we get the desired

13. sub-constraint:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \qquad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \Rightarrow \tau_1 <: c_2 \Rightarrow \tau_2}\ \text{sub-constraint}$$

To prove: $\forall (p, T, \Lambda.e) \in [\![(c_1 \Rightarrow \tau_1)\ \sigma\iota]\!].(p, T, \Lambda.e) \in [\![(c_2 \Rightarrow \tau_2)\ \sigma\iota]\!]$

This means given some $(p, T, \Lambda.e) \in [\![(c_1 \Rightarrow \tau_1)\ \sigma\iota]\!]$ we need to prove
$(p, T, \Lambda.e) \in [\![(c_2 \Rightarrow \tau_2)\ \sigma\iota]\!]$

From Definition 66 we are given that
$. \models c_1\iota \implies (p, T, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SC0)

Also from Definition 66 it suffices to prove that
$. \models c_2\iota \implies (p, T, e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$

This means given some $. \models c_2\iota$ and we need to prove
$(p, T, e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$ (F-SC1)

Since we are given that $\Theta; \Delta \models c_2 \implies c_1$ therefore we know that $. \models c_1\iota$

Hence from (F-SC0) we have
$(p, T, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SC2)

IH: $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_2)\ \sigma\iota]\!]$

Therefore we ge the desired from IH and (F-SC2)

14. sub-CAnd:

$$\frac{\Psi;\Theta;\Delta \vdash \tau_1 <: \tau_2 \qquad \Theta;\Delta \models c_1 \implies c_2}{\Psi;\Theta;\Delta \vdash c_1\&\tau_1 <: c_2\&\tau_2} \text{ sub-CAnd}$$

To prove: $\forall (p, v) \in [\![(c_1\&\tau_1)\ \sigma\iota]\!].(p, v) \in [\![(c_2\&\tau_2)\ \sigma\iota]\!]$

This means given some $(p, v) \in [\![(c_1\&\tau_1)\ \sigma\iota]\!]$ we need to prove
$(p, v) \in [\![(c_2\&\tau_2)\ \sigma\iota]\!]$

From Definition 66 we are given that
$. \models c_1\iota \wedge (p, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SCA0)

Also from Definition 66 it suffices to prove that
$. \models c_2\iota \wedge (p, e) \in [\![\tau_2\sigma\iota]\!]_{\mathcal{E}}$

Since we are given that $\Theta;\Delta \models c_2 \implies c_1$ and $. \models c_1\iota$ therefore we also know that $. \models c_2\iota$

Also from (F-SCA0) we have $(p, e) \in [\![\tau_1\sigma\iota]\!]_{\mathcal{E}}$ (F-SCA1)

<u>IH</u>: $[\![(\tau_1)\ \sigma\iota]\!] \subseteq [\![(\tau_2)\ \sigma\iota]\!]$

Therefore we ge the desired from IH and (F-SCA1)

15. sub-familyAbs:

$$\frac{\Psi;\Theta, i :S\vdash \tau <: \tau'}{\Psi;\Theta \vdash \lambda_t i :S .\tau <: \lambda_t i :S .\tau'} \text{ sub-familyAbs}$$

To prove:
$\forall f \in [\![\lambda_t i :S .\tau\ \sigma\iota]\!].f \in [\![\lambda_t i :S .\tau'\ \sigma\iota]\!]$

This means given $f \in [\![\lambda_t i :S .\tau\ \sigma\iota]\!]$ and we need to prove
$f \in [\![\lambda_t i :S .\tau'\ \sigma\iota]\!]$

This means from Definition 66 we are given
$\forall I.f\ I \in [\![\tau[I/i]\ \sigma\iota]\!]$ (F-SFAbs0)

This means from Definition 66 we need to prove
$\forall I'.f\ I' \in [\![\tau'[I'/i]\ \sigma\iota]\!]$
This further means that given some I' we need to prove
$f\ I' \in [\![\tau'[I'/i]\ \sigma\iota]\!]$ (F-SFAbs1)

Instantiating (F-SFAbs0) with I' we get
$f\ I' \in [\![\tau[I'/i]\ \sigma\iota]\!]$

From IH we know that $[\![\tau\ \sigma\iota \cup \{i \mapsto I'\ \iota\}]\!] \subseteq [\![\tau'\ \sigma\iota \cup \{i \mapsto I'\ \iota\}]\!]$
And this completes the proof.

16. Sub-tfamilyApp1:

$$\frac{}{\Psi; \Theta; \Delta \vdash \lambda_t i :S .\tau \ I <: \tau[I/i]} \text{ sub-familyApp1}$$

To prove:
$\forall (p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!].(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$ and we need to prove
$(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$

This means from Definition 66 we are given
$(p, T, v) \in [\![\lambda_t i :S .\tau]\!] \ I \ \sigma\iota$
This further means that we have
$(p, T, v) \in f \ I \ \sigma\iota$ where $f \ I \ \sigma\iota = [\![\tau[I/i] \ \sigma\iota]\!]$

This means we have $(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$
And this completes the proof.

17. Sub-tfamilyApp2:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: \lambda_t i :S .\tau \ I} \text{ sub-familyApp2}$$

To prove: $\forall (p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!].(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$

This means given $(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$ (Sub-tF0)
And we need to prove
$(p, T, v) \in [\![\lambda_t i :S .\tau \ I \ \sigma\iota]\!]$

This means from Definition 66 it suffices to prove that
$(p, T, v) \in [\![\lambda_t i :S .\tau]\!] \ I \ \sigma\iota$
It further suffices to prove that
$(p, T, v) \in f \ I \ \sigma\iota$ where $f \ I \ \sigma\iota = [\![\tau[I/i] \ \sigma\iota]\!]$

which means we need to show that
$(p, T, v) \in [\![\tau[I/i] \ \sigma\iota]\!]$
We get this directly from (Sub-tF0)

□

Lemma 73 (Expression subtyping lemma)**.** $\forall \Psi, \Theta, \tau, \tau'$.
$\Psi; \Theta \vdash \tau <: \tau' \implies [\![\tau \ \sigma\iota]\!]_{\mathcal{E}} \subseteq [\![\tau' \ \sigma\iota]\!]_{\mathcal{E}}$

Proof. To prove: $\forall (p, T, e) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}} \implies (p, T, e) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

This means given some $(p, T, e) \in [\![\tau\ \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, e) \in [\![\tau'\ \sigma\iota]\!]_{\mathcal{E}}$

This means from Definition 66 we are given
$\forall t < T, v.e \Downarrow_t v \implies (p, T - t, v) \in [\![\tau\ \sigma\iota]\!]$ (S-E0)

Similarly from Definition 66 it suffices to prove that
$\forall t' < T, v'.e \Downarrow_{t'} v' \implies (p, T - t, v') \in [\![\tau'\ \sigma\iota]\!]$

This means given some $t' < T, v'$ s.t $e \Downarrow_{t'} v'$ it suffices to prove that $(p, T - t', v') \in [\![\tau'\ \sigma\iota]\!]$

Instantiating (S-E0) with t', v' we get $(p, T - t', v') \in [\![\tau\ \sigma\iota]\!]$

And finally from Lemma 72 we get the desired. □

Lemma 74 (Γ subtyping lemma)**.** $\forall \Psi, \Theta, \Gamma_1, \Gamma_2, \sigma, \iota.$
$\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2 \implies [\![\Gamma_1 \sigma\iota]\!] \subseteq [\![\Gamma_2 \sigma\iota]\!]$

Proof. Proof by induction on $\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Gamma <: .}\ \text{sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in [\![\Gamma_1 \sigma\iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Gamma_1 \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![.]\!]_{\mathcal{E}}$

From Definition 67 it suffices to prove that
$\exists f : \mathcal{V}ars \to \mathcal{P}ots. (\forall x \in dom(.). (f(x), T, \gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}}) \wedge (\sum_{x \in dom(.)} f(x) \leqslant p)$

We choose f as a constant function $f' - = 0$ and we get the desired

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta \vdash \tau' <: \tau \qquad \Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2, x : \tau}\ \text{sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in [\![\Gamma_1 \sigma\iota]\!]_{\mathcal{E}}.(p, T, \gamma) \in [\![\Gamma_2, x : \tau]\!]_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in [\![\Gamma_1 \sigma\iota]\!]_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in [\![\Gamma_2, x : \tau]\!]_{\mathcal{E}}$

This means from Definition 67 we are given that
$\exists f : \mathcal{V}ars \to \mathcal{P}ots.$
$(\forall x \in dom(\Gamma_1). (f(x), T, \gamma(x)) \in [\![\Gamma(x)]\!]_{\mathcal{E}})$ (L0)
$(\sum_{x \in dom(\Gamma_1)} f(x) \leqslant p)$ (L1)

Similarly from Definition 67 it suffices to prove that
$\exists f' : \mathcal{V}ars \to \mathcal{P}ots. (\forall y \in dom(\Gamma_2, x : \tau). (f'(y), T, \gamma(y)) \in [\![(\Gamma_2, x : \tau)(y)]\!]_{\mathcal{E}}) \wedge (\sum_{y \in dom(\Gamma_2, x:\tau)} f'(y) \leqslant p)$

We choose f' as f and it suffices to prove that

(a) $\forall y \in dom(\Gamma_2, x : \tau). (f(y), T, \gamma(y)) \in \llbracket(\Gamma_2, x : \tau)(y)\rrbracket_{\mathcal{E}}$:
This means given some $y \in dom(\Gamma_2, x : \tau)$ it suffices to prove that $(f(y), T, \gamma(y)) \in \llbracket\tau_2\rrbracket_{\mathcal{E}}$ where say $(\Gamma_2, x : \tau)(y) = \tau_2$

From Lemma 75 we know that
$y : \tau_1 \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau_1 <: \tau_2$

By instantiating (L0) with the given y
$(f(y), T, \gamma(y)) \in \llbracket\tau_1\rrbracket_{\mathcal{E}}$

Finally from Lemma 73 we also get $(f(y), T, \gamma(y)) \in \llbracket\tau_2\rrbracket_{\mathcal{E}}$
And we are done

(b) $(\sum_{y \in dom(\Gamma_2, x:\tau)} f(y) \leqslant p)$:
From (L1) we know that $(\sum_{x \in dom(\Gamma_1)} f(x) \leqslant p)$ and since from Lemma 75 we know that $dom(\Gamma_2, x : \tau) \subseteq dom(\Gamma_1)$ therefore we also have
$(\sum_{y \in dom(\Gamma_2, x:\tau)} f(y) \leqslant p)$

□

Lemma 75 (Γ Subtyping: domain containment)**.** $\forall p, \gamma, \Gamma_1, \Gamma_2.$
$\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2 \implies \forall x : \tau \in \Gamma_2.\ x : \tau' \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Proof. Proof by induction on $\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Gamma_1 <: .}\ \text{sub-lBase}$$

To prove: $\forall x : \tau' \in (.).x : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Trivial

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \qquad \Psi; \Theta \vdash \tau' <: \tau \qquad \Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2, x : \tau_x}\ \text{sub-lBase}$$

To prove: $\forall y : \tau_1 \in (\Gamma_2, x : \tau_x).y : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

This means given some $y : \tau \in (\Gamma_2, x : \tau_x)$ it suffices to prove that
$y : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

The follwing cases arise:

- $y = x$:
In this case we are given that $x : \tau' \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$
Therefore we are done
- $y \neq x$:
Since we are given that $\Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2$ therefore we get the desired from IH

$\square$

Lemma 76 (Ω subtyping lemma)**.** $\forall \Psi, \Theta, \Omega_1, \Omega_2, \sigma, \iota.$
$\Psi; \Theta \vdash \Omega_1 <: \Omega_2 \implies [\![\Omega_1 \sigma \iota]\!] \subseteq [\![\Omega_2 \sigma \iota]\!]$

Proof. Proof by induction on $\Psi; \Theta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Omega <: .} \text{ sub-mBase}$$

   To prove: $\forall (0, T, \delta) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}.(0, T, \delta) \in [\![.]\!]_{\mathcal{E}}$

   This means given some $(0, T, \delta) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(0, T, \delta) \in [\![.]\!]_{\mathcal{E}}$

   We get the desired directly from Definition 67

2. sub-lInd:

$$\frac{x : \tau' \in \Omega_1 \qquad \Psi; \Theta \vdash \tau' <: \tau \qquad \Psi; \Theta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta \vdash \Omega_1 <: \Omega_2, x : \tau} \text{ sub-mInd}$$

   To prove: $\forall (0, T, \delta) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}.(0, T, \delta) \in [\![\Omega_2, x : \tau]\!]_{\mathcal{E}}$

   This means given some $(0, T, \delta) \in [\![\Omega_1 \sigma \iota]\!]_{\mathcal{E}}$ it suffices to prove that $(0, T, \delta) \in [\![\Omega_2, x : \tau]\!]_{\mathcal{E}}$

   This means from Definition 67 we are given that
   $(\forall x : \tau \in \Omega_1. (0, T, \delta(x)) \in [\![\tau]\!]_{\mathcal{E}})$ (L0)

   Similarly from Definition 67 it suffices to prove that
   $(\forall y : \tau_y \in (\Omega_2, x : \tau).(0, T, \delta(y)) \in [\![\tau_y]\!]_{\mathcal{E}})$

   This means given some $y : \tau_y \in (\Omega_2, x : \tau)$ it suffices to prove that
   $(0, T, \delta(y)) \in [\![\tau_y]\!]_{\mathcal{E}}$

   From Lemma 77 we know that $\exists \tau'.y : \tau' \in dom(\Omega_1) \wedge \Psi; \Theta \vdash \tau' <: \tau_y$

   Instantiating (L0) with $y : \tau'$ we get $(0, T, \delta(y)) \in [\![\tau']\!]_{\mathcal{E}}$

   And finally from Lemma 73 we get the desired

$\square$

Lemma 77 (Ω Subtyping: domain containment)**.** $\forall \Psi, \Theta, \Omega_1, \Omega_2.$
$\Psi; \Theta \vdash \Omega_1 <: \Omega_2 \implies \forall x : \tau \in \Omega_2.\ x : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Proof. Proof by induction on $\Psi; \Theta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Omega <: .} \text{ sub-mBase}$$

   To prove: $\forall x : \tau \in (.).x : \tau' \in \Omega \wedge \Psi; \Theta \vdash \tau' <: \tau$

   Trivial

2. sub-lInd:

$$\frac{x : \tau' \in \Omega_1 \qquad \Psi; \Theta \vdash \tau' <: \tau \qquad \Psi; \Theta \vdash \Omega_1 / x <: \Omega_2}{\Psi; \Theta \vdash \Omega_1 <: \Omega_2, x : \tau} \text{ sub-mInd}$$

To prove: $\forall y : \tau \in (\Omega_2, x : \tau_x). y : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

This means given some $y : \tau \in (\Omega_2, x : \tau)$ it suffices to prove that
$y : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

The following cases arise:

- $y = x$:
  In this case we are given that
  $x : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$
  Therefore we are done
- $y \neq x$:
  Since we are given that $\Psi; \Theta \vdash \Omega_1 / x <: \Omega_2$ therefore we get the desired from IH

□

Theorem 78 (Soundness 1)**.** $\forall e, n, n', \tau \in Type, t.$
$\vdash e : \mathbb{M}\, n\, \tau \wedge e \Downarrow_t^{n'} v \implies n' \leqslant n$

Proof. From Theorem 71 we know that $(0, t+1, e) \in [\![\mathbb{M}\, n\, \tau]\!]_{\mathcal{E}}$

From Definition 66 this means we have
$\forall t' < t + 1. e \Downarrow_{t'} v' \implies (0, t + 1 - t'v') \in [\![\mathbb{M}\, n\, \tau]\!]$

From the evaluation relation we know that $e \Downarrow_0 e$ therefore we have
$(0, t + 1, e) \in [\![\mathbb{M}\, n\, \tau]\!]$

Again from Definition 66 it means we have
$\forall t'' < t + 1. e \Downarrow_{t'}^{n'} v \implies \exists p'. n' + p' \leqslant 0 + n \wedge (p', t + 1 - t'', v) \in [\![\tau]\!]$

Since we are given that $e \Downarrow_t^{n'} v$ therefore we have
$\exists p'. n' + p' \leqslant n \wedge (p', 1, v) \in [\![\tau]\!]$

Since $p' \geqslant 0$ therefore we get $n' \leqslant n$

□

Theorem 79 (Soundness 2)**.** $\forall e, n, n', \tau \in Type.$
$\vdash e : [n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau \wedge e\, () \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v \implies n' \leqslant n$

Proof. From Theorem 71 we know that $(0, t_1 + t_2 + 2, e) \in [\![[n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau]\!]_{\mathcal{E}}$

Therefore from Definition 66 we know that
$$\forall t' < t_1 + t_2 + 2, v. e \Downarrow_{t'} v \implies (0, t_1 + t_2 + 2 - t', v) \in [\![[n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau]\!] \tag{S0}$$

Since we know that $e\, () \Downarrow_{t_1} -$ therefore from E-app we know that $\exists e'. e \Downarrow_{t_1} \lambda x. e'$

Instantiating (S0) with $t_1, \lambda x. e'$ we get $(0, t_2 + 2, \lambda x. e') \in [\![[n]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \tau]\!]$

This means from Definition 66 we have
$$\forall p', e', t'' < t_2 + 2. (p', t'', e'') \in [\![[n]\, \mathbf{1}]\!]_{\mathcal{E}} \implies (0 + p', t'', e'[e''/x]) \in [\![\mathbb{M}\, 0\, \tau]\!]_{\mathcal{E}} \tag{S1}$$

Claim: $\forall t.(I, t, ()) \in [\![[I]\,\mathbf{1}]\!]_{\mathcal{E}}$
Proof:
From Definition 66 it suffices to prove that
$() \Downarrow_0 v \implies (I, t, v) \in [\![[I]\,\mathbf{1}]\!]$

Since we know that $v = ()$ therefore it suffices to prove that
$(I, t, v) \in [\![[I]\,\mathbf{1}]\!]$

From Definition 66 it suffices to prove that
$\exists p'.p' + I \leqslant I \wedge (p', t, v) \in [\![\mathbf{1}]\!]\}$

We choose p' as 0 and we get the desired

Instantiating (S1) with $n, (), t_2 + 1$ we get $(n, t_2 + 1, e'[()/x]) \in [\![\mathbb{M}\,0\,\tau]\!]_{\mathcal{E}}$

This means again from Definition 66 we have
$\forall t' < t_2 + 1.e'[()/x] \Downarrow_{t'} v' \implies (n, t_2 + 1 - t', v') \in [\![\mathbb{M}\,0\,\tau]\!]$

From E-val we know that $v' = e'[()/x]$ and $t' = 0$ therefore we have
$(n, t_2 + 1, e'[()/x]) \in [\![\mathbb{M}\,0\,\tau]\!]$

Again from Definition 66 we have
$\forall t' < t_2 + 1.e'[()/x] \Downarrow_{t'}^{n'} v'' \implies \exists p'.n' + p' \leqslant n + 0 \wedge (p', t_2 + 1 - t', v'') \in [\![\tau]\!]$

Since we are given that $e \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v$ therefore we get
$\exists p'.n' + p' \leqslant n \wedge (p', 1, v'') \in [\![\tau]\!]$

Since $p' \geqslant 0$ therefore we have $n' \leqslant n$

□

Corollary 80 (Soundness)**.** $\forall \Gamma, e, q, q', \tau, T, p_l.$
$.; .; .; \Gamma \vdash e : [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,[q']\,\tau \;\wedge$
$(p_l, T, \gamma) \in [\![\Gamma]\!]_{\mathcal{E}} \;\wedge$
$e\;()\;\gamma \Downarrow_{t_1} v_t \Downarrow_{t_2}^{J} v \;\wedge$
$t_1 + t_2 < T$
$\implies$
$\exists p_v.(p_v, T - t_1 - t_2, v) \in [\![\tau]\!] \wedge J \leqslant (q + p_l) - (q' + p_v)$

Proof. From Theorem 71 we know that $(p_l, T, e) \in [\![[q]\,\mathbf{1} \multimap \mathbb{M}\,0\,[q']\,\tau]\!]_{\mathcal{E}}$

Therefore from Definition 66 we know that
$\forall\, T' < T, v.e\gamma \Downarrow_{T'} v \implies (p_l, T - T', v) \in [\![[q]\,\mathbf{1} \multimap \mathbb{M}\,0\,[q']\,\tau]\!]$ (S0)

Since we know that $e\;()\gamma \Downarrow_{t_1} v_t$ therefore from E-app we know that
$\exists e'.e \Downarrow_{t_1'} \lambda x.e'$ and $e'[()/x] \Downarrow_{t_1''} v_t$ s.t $t_1' + t_1'' + 1 = t_1$

Instantiating (S0) with $t_1', \lambda x.e'$ we get $(p_l, T - t_1', \lambda x.e') \in [\![[q]\,\mathbf{1} \multimap \mathbb{M}\,0\,[q']\,\tau]\!]$

This means from Definition 66 we have
$\forall p', T' < (T - t_1'), e'.(p', T', e'') \in [\![[q]\,\mathbf{1}]\!]_{\mathcal{E}} \implies (p_l + p', T', e'[e''/x]) \in [\![\mathbb{M}\,0\,[q']\,\tau]\!]_{\mathcal{E}}$ (S1)

Claim: $\forall\, T\,.(I, T, ()) \in [\![[I]\,\mathbf{1}]\!]_{\mathcal{E}}$
Proof:
From Definition 66 it suffices to prove that
$\forall\, T'' < T, v.() \Downarrow_{T''} v \implies (I, T - T'', v) \in [\![[I]\,\mathbf{1}]\!]$

From (E-val) we know that $T'' = 0$ and $v = ()$ therefore it suffices to prove that

$(I, T, ()) \in [\![[I]\,\mathbf{1}]\!]$

From Definition 66 it further suffices to prove that
$\exists p'.p' + I \leqslant I \wedge (p', T, ()) \in [\![\mathbf{1}]\!]\}$

We choose p' as 0 and we get the desired

$\square$

Using the claim we know that we have $(q, T - t_1' - 1, ()) \in [\![[q]\,\mathbf{1}]\!]_{\mathcal{E}}$
Instantiating (S1) with $q, T - t_1' - 1, ()$ and using the claim proved above we get
$(p_l + q, T - t_1' - 1, e'[()/x]) \in [\![\mathbb{M}\,0\,[q']\,\tau]\!]_{\mathcal{E}}$

This means again from Definition 66 we have
$\forall\, T_1 < T - t_1' - 1.e'[()/x] \Downarrow v' \implies (p_l + q, T - t_1' - 1 - T_1, v') \in [\![\mathbb{M}\,0\,[q']\,\tau]\!]$

Instantiating with t_1'', v_t and since $t_1 < T$, therefore we also have $t_1'' < T - t_1'$.
Also since we are given that $e()\gamma \Downarrow_{t_1} v_t$, therefore we know that $v' = v_t$. Thus, we have
$(p_l + q, T - t_1' - 1 - t_1'', v_t) \in [\![\mathbb{M}\,0\,[q']\,\tau]\!]$

Again from Definition 66 we have
$\forall v'', t_2' < T - t_1' - t_1'' - 1.v_t \Downarrow_{t_2'}^{J} v'' \implies \exists p'.J + p' \leqslant p_l + q \wedge (p', T - t_1' - t_1'' - 1 - t_2', v'') \in [\![[q']\,\tau]\!]$

Instantiating with v, t_2 and since $t_2 < T - t_1' - t_1'' - 1$ and $e \Downarrow_{t_1} v_t \Downarrow_{t_2}^{J} v$ therefore we get
$\exists p'.J + p' \leqslant p_l + q \wedge (p', T - t_1' - t_1'' - 1 - t_2, v) \in [\![[q']\,\tau]\!]$ (S2)

Since we have $(p', T - t_1' - t_1'' - 1 - t_2, v) \in [\![[q']\,\tau]\!]$ therefore from Definition 66 we have
$\exists p_1'.p_1' + q' \leqslant p' \wedge (p_1', T - t_1' - t_1'' - 1 - t_2, v) \in [\![\tau]\!]\}$ (S3)

In order to prove $\exists p_v.(p_v, T - t_1 - t_2, v) \in [\![\tau]\!] \wedge J \leqslant (q + p_l) - (q' + p_v)$ we choose p_v as p_1' and we need to prove

1. $(p_1', T - t_1 - t_2, v) \in [\![\tau]\!]$:
   Since from (S3) we have $(p_1', T - t_1' - t_1'' - 1 - t_2, v) \in [\![\tau]\!]$ and since $t_1' + t_1'' + 1 = t_1$ therefore also have
   $(p_1', T - t_1 - t_2, v) \in [\![\tau]\!]$
2. $J \leqslant (q + p_l) - (q' + p_v)$:
   From (S2) and (S3) we get
   $J \leqslant (p_l + q) - (q' + p_1')$

$\square$

2.5 Embedding Univariate RAML

Univariate RAML's type syntax

$$\begin{array}{lrcl} \text{Types} & \tau & ::= & \mathsf{b} \mid L^{\vec{q}}\,\tau \mid (\tau_1, \tau_2) \\ & A & ::= & \tau \xrightarrow{q/q'} \tau \end{array}$$

Type translation

$$\begin{array}{lcl} (\!|unit|\!) & = & \mathbf{1} \\ (\!|\mathsf{b}|\!) & = & !\mathsf{b} \\ (\!|L^{\vec{q}}\,\tau|\!) & = & \exists s.([\phi(\vec{q}, s)]\,\mathbf{1} \otimes L^{s}(\!|\tau|\!)) \\ (\!|(\tau_1, \tau_2)|\!) & = & ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \\ (\!|\tau_1 \xrightarrow{q/q'} \tau_2|\!) & = & ([q]\,\mathbf{1} \multimap (\!|\tau_1|\!) \multimap \mathbb{M}\,0\,[q']\,(\!|\tau_2|\!)) \end{array}$$

Type context translation

$$\begin{array}{rcl} (\!|.|\!) & = & . \\ (\!|\Gamma, x:\tau|\!) & = & (\!|\Gamma|\!), x : (\!|\tau|\!) \end{array}$$

Function context translation

$$\begin{array}{rcl} (\!|.|\!) & = & . \\ (\!|\Sigma, x:\tau|\!) & = & (\!|\Sigma|\!), x : (\!|\tau|\!) \end{array}$$

Judgment translation

$$\boxed{\Sigma; \Gamma \vdash_{q'}^{q} e_r : \tau \quad \rightsquigarrow \quad .; .; (\!|\Sigma|\!); (\!|\Gamma|\!) \vdash e_a : [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q'](\!|\tau|\!))}$$

Definition 81. $\phi(\vec{q}, n) \triangleq \sum_{1 \leqslant i \leqslant k} \binom{n}{i} q_i$ as defined in [2, 1]

Expression translation

$$\frac{}{\Sigma; . \vdash_{q}^{q+K^{unit}} () : unit \rightsquigarrow \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{unit}} \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{ret}(a)} \text{ unit}$$

$$\frac{}{\Sigma; . \vdash_{q}^{q+K^{base}} c : \mathsf{b} \rightsquigarrow \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{base}} \text{ in } \mathsf{bind}\, a = \mathsf{store}(!c) \text{ in } \mathsf{ret}(a)} \text{ base}$$

$$\frac{}{\Sigma; x : \tau \vdash_{q}^{q+K^{var}} x : \tau \rightsquigarrow \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a)} \text{ var}$$

$$\frac{\tau_1 \xrightarrow{q/q'} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash_{q'-K_2^{app}}^{q+K_1^{app}} f\ x : \tau_2 \rightsquigarrow \lambda u.E_0} \text{ app}$$

where
$E_0 = \mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K_1^{app}} \text{ in } \mathsf{bind}\, P = \mathsf{store}() \text{ in } E_1$
$E_1 = \mathsf{bind}\, f_1 = (f\ P\ x) \text{ in } \mathsf{release}\, f_2 = f_1 \text{ in } \mathsf{bind} - = \uparrow^{K_2^{app}} \text{ in } \mathsf{bind}\, f_3 = \mathsf{store}\, f_2 \text{ in } \mathsf{ret}\, f_3$

$$\frac{}{\begin{array}{c}\Sigma; \varnothing \vdash_{q}^{q+K^{nil}} nil : L^{\vec{p}}\tau \rightsquigarrow \\ \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{nil}} \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil\rangle\!\rangle \text{ in } \mathsf{ret}(b)\end{array}} \text{ nil}$$

$$\frac{\vec{p} = (p_1, \ldots, p_k)}{\begin{array}{c}\Sigma; x_h : \tau, x_t : L^{(\lhd\ \vec{p})}\tau \vdash_{q}^{q+p_1+K^{cons}} cons(x_h, x_t) : L^{p}\tau \rightsquigarrow \\ \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{cons}} \text{ in } E_0\end{array}} \text{ cons}$$

where
$E_0 = x_t; x.\, \mathsf{let}\langle\!\langle x_1, x_2\rangle\!\rangle = x \text{ in } E_1$
$E_1 = \mathsf{release} - = x_1 \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, x_h :: x_2\rangle\!\rangle \text{ in } \mathsf{ret}(b)$

$$\frac{\begin{array}{c}\Sigma; \Gamma \vdash_{q'+K_2^{matN}}^{q-K_1^{matN}} e_1 : \tau' \rightsquigarrow e_{a1} \\ \vec{p} = (p_1, \ldots, p_k) \qquad \Sigma; \Gamma, h : \tau, t : L^{(\lhd\ \vec{p})}\tau \vdash_{q'+K_2^{matN}}^{q+p_1-K_1^{matC}} e_2 : \tau' \rightsquigarrow e_{a2}\end{array}}{\Sigma; \Gamma; x : L^{p}\tau \vdash_{q'}^{q} \mathsf{match}\ x\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2 : \tau' \rightsquigarrow \qquad \lambda u.E_0} \text{ match}$$

where
$E_0 = \mathsf{release} - = u \mathsf{~in~} E_{0.1}$
$E_{0.1} = x; a. \mathsf{let}\langle\langle x_1, x_2\rangle\rangle = a \mathsf{~in~} E_1$
$E_1 = \mathsf{match~} x_2 \mathsf{~with~} |nil \mapsto E_2 \; |h :: l_t \mapsto E_3$
$E_2 = \mathsf{bind} - = \uparrow^{K_1^{matN}} \mathsf{~in~} E_{2.1}$
$E_{2.1} = \mathsf{bind}\, b = \mathsf{store}() \mathsf{~in~} E_2'$
$E_2' = \mathsf{bind}\, c = (e_{a1}\; b) \mathsf{~in~} E_{2.1}'$
$E_{2.1}' = \mathsf{release}\, d = c \mathsf{~in~} E_{2.2}'$
$E_{2.2}' = \mathsf{bind} - = \uparrow^{K_2^{matN}} \mathsf{~in~} E_{2.3}'$
$E_{2.3}' = \mathsf{release} - = x_1 \mathsf{~in~store}\, d$
$E_3 = \mathsf{bind} - = \uparrow^{K_1^{matC}} \mathsf{~in~} E_{3.1}$
$E_{3.1} = \mathsf{release} - = x_1 \mathsf{~in~} E_{3.2}$
$E_{3.2} = \mathsf{bind}\, b = \mathsf{store}() \mathsf{~in~} E_{3.3}$
$E_{3.3} = \mathsf{bind}\, t = \mathsf{ret}\langle\langle b, l_t\rangle\rangle \mathsf{~in~} E_{3.4}$
$E_{3.4} = \mathsf{bind}\, d = \mathsf{store}() \mathsf{~in~} E_{3.5}$
$E_{3.5} = \mathsf{bind}\, f = e_{a2}\; d \mathsf{~in~} E_{3.6}$
$E_{3.6} = \mathsf{release}\, g = f \mathsf{~in~} E_{3.7}$
$E_{3.7} = \mathsf{bind} - = \uparrow^{K_2^{matC}} \mathsf{~in~store}\, g$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^q_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma; \Gamma, z : \tau \vdash^q_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-unit}$$

$E_0 = \lambda u. E_1$
$E_1 = \mathsf{bind}\, a = coerce_{\mathbf{1},\mathbf{1},\mathbf{1}}\; z \mathsf{~in~let}\langle\langle x, y\rangle\rangle = a \mathsf{~in~} e_a\; u$

$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} : (\!|\mathbf{1}|\!) \multimap \mathbb{M}\, 0\, ((\!|\mathbf{1}|\!) \otimes (\!|\mathbf{1}|\!))$
$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} \triangleq \lambda u.\, \mathsf{ret}\langle\langle !(), !()\rangle\rangle$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^q_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathsf{b}}{\Sigma; \Gamma, z : \tau \vdash^q_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-base}$$

$E_0 = \lambda u. E_1$
$E_1 = \mathsf{bind}\, a = coerce_{\mathsf{b},\mathsf{b},\mathsf{b}}\; z \mathsf{~in~let}\langle\langle x, y\rangle\rangle = a \mathsf{~in~} e_a\; u$

$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} : (\!|\mathsf{b}|\!) \multimap \mathbb{M}\, 0\, ((\!|\mathsf{b}|\!) \otimes (\!|\mathsf{b}|\!))$
$coerce_{\mathsf{b},\mathsf{b},\mathsf{b}} \triangleq \lambda u.\, \mathsf{let}\, !\, u' = u \mathsf{~in~ret}\langle\langle !u', !u'\rangle\rangle$

$$\frac{\begin{array}{c}\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^q_{q'} e : \tau' \rightsquigarrow e_a \\ \tau = L^{\vec{p}}\tau'' \qquad \tau_1 = L^{\vec{p_1}}\tau_1'' \qquad \tau_2 = L^{\vec{p_2}}\tau_2'' \qquad \tau'' = \tau_1'' \curlyvee \tau_2'' \qquad \vec{p} = \vec{p_1} + \vec{p_2}\end{array}}{\Sigma; \Gamma, z : \tau \vdash^q_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-list}$$

$E_0 = \lambda u. E_1$
$E_1 = \mathsf{bind}\, a = coerce_{\tau,\tau_1,\tau_2}\; z \mathsf{~in~let}\langle\langle x, y\rangle\rangle = a \mathsf{~in~} e_a\; u$

$coerce_{L^{\vec{p}}\tau, L^{\vec{p_1}}\tau_1, L^{\vec{p_2}}\tau_2} :!((\!|\tau|\!) \multimap \mathbb{M}\, 0\, (\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \multimap (\!|L^{\vec{p}}\tau|\!) \multimap \mathbb{M}\, 0\, (\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!)$
$coerce_{L^{\vec{p}}\tau, L^{\vec{p_1}}\tau_1, L^{\vec{p_2}}\tau_2} \triangleq \mathsf{fix} f. \lambda\_g. \lambda e.\, \mathsf{let}\, !\, g' = g \mathsf{~in~} e; x.\, \mathsf{let}\langle\langle p, l\rangle\rangle = x \mathsf{~in~} E_0$

where

$E_0 \triangleq \mathsf{release} - = p \mathsf{~in~} E_1$
$E_1 \triangleq \mathsf{match~} l \mathsf{~with~} |nil \mapsto E_{2.1} \ |h :: t \mapsto E_3$
$E_{2.1} \triangleq \mathsf{bind~} z_1 = \mathsf{store}() \mathsf{~in~} E_{2.2}$
$E_{2.2} \triangleq \mathsf{bind~} z_2 = \mathsf{store}() \mathsf{~in~} E_{2.3}$
$E_{2.3} \triangleq \mathsf{ret}\langle\langle\langle\langle z_1, nil\rangle\rangle, \langle\langle z_2, nil\rangle\rangle\rangle\rangle$
$E_3 \triangleq \mathsf{bind~} H = g' \ h \mathsf{~in~} E_{3.1}$
$E_{3.1} \triangleq \mathsf{bind~} o_t = () \mathsf{~in~} E_{3.2}$
$E_{3.2} \triangleq \mathsf{bind~} T = f \ g \ \langle\langle o_t, t\rangle\rangle \mathsf{~in~} E_4$
$E_4 \triangleq \mathsf{let}\langle\langle H_1, H_2\rangle\rangle = H \mathsf{~in~} E_5$
$E_5 \triangleq \mathsf{let}\langle\langle T_1, T_2\rangle\rangle = T \mathsf{~in~} E_6$
$E_6 \triangleq T_1; tp_1. \mathsf{let}\langle\langle p_1', l_1'\rangle\rangle = tp_1 \mathsf{~in~} E_{7.1}$
$E_{7.1} \triangleq T_2; tp_2. \mathsf{let}\langle\langle p_2', l_2'\rangle\rangle = tp_2 \mathsf{~in~} E_{7.2}$
$E_{7.2} \triangleq \mathsf{release} - = p_1' \mathsf{~in~} E_{7.3}$
$E_{7.3} \triangleq \mathsf{release} - = p_2' \mathsf{~in~} E_{7.4}$
$E_{7.4} \triangleq \mathsf{bind~} o_1 = \mathsf{store}() \mathsf{~in~} E_{7.5}$
$E_{7.5} \triangleq \mathsf{bind~} o_2 = \mathsf{store}() \mathsf{~in~} E_8$
$E_8 \triangleq \mathsf{ret}\langle\langle\langle\langle o_1, H_1 :: T_1\rangle\rangle, \langle\langle o_2, H_2 :: T_2\rangle\rangle\rangle\rangle$

$$\frac{\begin{array}{c}\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^q_{q'} e : \tau' \rightsquigarrow e_a \\ \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = (\tau_a, \tau_b) \qquad \tau_1 = (\tau_a', \tau_b') \qquad \tau_2 = (\tau_a'', \tau_b'')\end{array}}{\Sigma; \Gamma, z : \tau \vdash^q_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-pair}$$

$E_0 = \lambda u.E_1$
$E_1 = \mathsf{bind~} a = coerce_{(\tau_a,\tau_b),(\tau_a',\tau_b'),(\tau_a'',\tau_b'')} \ z \mathsf{~in~} \mathsf{let}\langle\langle x, y\rangle\rangle = a \mathsf{~in~} e_a \ u$

$coerce_{(\tau_a,\tau_b),(\tau_a',\tau_b'),(\tau_a'',\tau_b'')} :!(\llparenthesis\tau_a\rrparenthesis \multimap \mathbb{M}\,0\,\llparenthesis\tau_a'\rrparenthesis \otimes \llparenthesis\tau_a''\rrparenthesis) \multimap !(\llparenthesis\tau_b\rrparenthesis \multimap \mathbb{M}\,0\,\llparenthesis\tau_b'\rrparenthesis \otimes \llparenthesis\tau_b''\rrparenthesis) \multimap \llparenthesis(\tau_a, \tau_b)\rrparenthesis \multimap \mathbb{M}\,0\,\llparenthesis(\tau_a', \tau_b')\rrparenthesis \otimes \llparenthesis(\tau_a'', \tau_b'')\rrparenthesis$
$coerce_{(\tau_a,\tau_b),(\tau_a',\tau_b'),(\tau_a'',\tau_b'')} \triangleq \lambda\_g_1.\lambda\_g_2.\lambda p. \mathsf{let}\,!\langle\langle p_1, p_2\rangle\rangle = p \mathsf{~in~} E_0$
where
$E_0 \triangleq \mathsf{let}\,!\,g_1' = g_1 \mathsf{~in~} E_1$
$E_1 \triangleq \mathsf{let}\,!\,g_2' = g_2 \mathsf{~in~} E_2$
$E_2 \triangleq \mathsf{bind~} P_1' = g_1' p_1 \mathsf{~in~} E_3$
$E_3 \triangleq \mathsf{bind~} P_2' = g_2' p_2 \mathsf{~in~} E_4$
$E_4 \triangleq \mathsf{let}\,!\langle\langle p_{11}', p_{12}'\rangle\rangle = P_1' \mathsf{~in~} E_5$
$E_5 \triangleq \mathsf{let}\,!\langle\langle p_{21}', p_{22}'\rangle\rangle = P_2' \mathsf{~in~} E_6$
$E_6 \triangleq \mathsf{ret}\langle\langle p_{11}', p_{21}'\rangle\rangle, \langle\langle p_{12}', p_{22}'\rangle\rangle$

$$\frac{\Sigma; \Gamma \vdash^q_{q'} e : \tau \rightsquigarrow e_a \qquad \tau <: \tau'}{\Sigma; \Gamma \vdash^q_{q'} e : \tau' \rightsquigarrow e_a} \text{ Sub}$$

$$\frac{\Sigma; \Gamma, x : \tau_1 \vdash^q_{q'} e : \tau \rightsquigarrow e_a \qquad \tau_1' <: \tau_1}{\Sigma; \Gamma, x : \tau_1' \vdash^q_{q'} e : \tau \rightsquigarrow e_a} \text{ Super}$$

$$\frac{\Sigma; \Gamma \vdash^p_{p'} e : \tau \rightsquigarrow e_a \qquad q \geqslant p \qquad q - p \geqslant q' - p'}{\Sigma; \Gamma \vdash^q_{q'} e : \tau \rightsquigarrow \lambda o.E_0} \text{ Relax}$$

where
$E_0 = \mathsf{release} - = o \mathsf{~in~} E_1$

$E_1 = \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ E_2$
$E_2 = \mathsf{bind}\, b = e_a\ a\ \mathsf{in}\ E_3$
$E_3 = \mathsf{release}\, c = b\ \mathsf{in}\ \mathsf{store}\, c$

$$\frac{\Sigma; \Gamma_1 \vdash_{p}^{q-K_1^{let}} e_1 : \tau_1 \rightsquigarrow e_{a1} \qquad \Sigma; \Gamma_2, x : \tau_1 \vdash_{q'+K_3^{let}}^{p-K_2^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma; \Gamma_1, \Gamma_2 \vdash_{q'}^{q} \mathsf{let}\ x = e_1\ in\ e_2 : \tau \rightsquigarrow E_t} \text{ Let}$$

where
$E_t = \lambda u.E_0$
$E_0 = \mathsf{release} - = u\ \mathsf{in}\ E_1$
$E_1 = \mathsf{bind} - = \uparrow^{K_1^{let}}\ \mathsf{in}\ E_2$
$E_2 = \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ E_3$
$E_3 = \mathsf{bind}\, b = e_{a1}\ a\ \mathsf{in}\ E_4$
$E_4 = \mathsf{release}\, x = b\ \mathsf{in}\ E_5$
$E_5 = \mathsf{bind} - = \uparrow^{K_2^{let}}\ \mathsf{in}\ E_6$
$E_6 = \mathsf{bind}\, c = \mathsf{store}()\ \mathsf{in}\ E_7$
$E_7 = \mathsf{bind}\, d = e_{a2}\ c\ \mathsf{in}\ E_8$
$E_8 = \mathsf{release}\, f = d\ \mathsf{in}\ E_9$
$E_9 = \mathsf{bind} - = \uparrow^{K_3^{let}}\ \mathsf{in}\ E_{10}$
$E_{10} = \mathsf{bind}\, g = \mathsf{store}\, f\ \mathsf{in}\ \mathsf{ret}\, g$

$$\frac{}{\Sigma; x_1 : \tau_1, x_2 : \tau_2 \vdash_{q}^{q+K^{pair}} (x_1, x_2) : (\tau_1, \tau_2) \rightsquigarrow E_t} \text{ pair}$$

where
$E_t = \lambda u.E_0$
$E_0 = \mathsf{release} - = u\ \mathsf{in}\ E_1$
$E_1 = \mathsf{bind} - = \uparrow^{K^{pair}}\ \mathsf{in}\ E_2$
$E_2 = \mathsf{bind}\, a = \mathsf{store}(x_1, x_2)\ \mathsf{in}\ \mathsf{ret}\, a$

$$\frac{\tau = (\tau_1, \tau_2) \qquad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash_{q'+K_2^{matP}}^{q-K_1^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash_{q}^{q} match\ x\ with\ (x_1, x_2) \to e : \tau' \rightsquigarrow E_t} \text{ matP}$$

where
$E_t = \lambda u.E_0$
$E_0 = \mathsf{release} - = u\ \mathsf{in}\ E_1$
$E_1 = \mathsf{bind} - = \uparrow^{K_1^{matP}}\ \mathsf{in}\ E_2$
$E_2 = \mathsf{let}\langle\langle x_1, x_2 \rangle\rangle = x\ \mathsf{in}\ E_3$
$E_3 = \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ E_4$
$E_4 = \mathsf{bind}\, b = e_t\ a\ \mathsf{in}\ E_5$
$E_5 = \mathsf{release}\, c = b\ \mathsf{in}\ E_6$
$E_6 = \mathsf{bind} - = \uparrow^{K_2^{matP}}\ \mathsf{in}\ E_7$
$E_7 = \mathsf{bind}\, d = \mathsf{store}\, c\ \mathsf{in}\ \mathsf{ret}\, d$

$$\frac{\Sigma; \Gamma \vdash_{q'}^{q} e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash_{q'}^{q} e : \tau \rightsquigarrow e_a} \text{ Augment}$$

2.5.1 Type preservation

Theorem 82 (Type preservation: Univariate RAML to λ-Amor). If $\Sigma; \Gamma \vdash^q_{q'} e : \tau$ in Univariate RAML then there exists e' such that $\Sigma; \Gamma \vdash^q_{q'} e : \tau \rightsquigarrow e'$ such that there is a derivation of $.; .; .; (\!|\Sigma|\!), (\!|\Gamma|\!) \vdash e' : [q]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q'] (\!|\tau|\!))$ in λ-Amor .

Proof. By induction on $\Sigma; \Gamma \vdash^q_{q'} e : \tau$

1. unit:

$$\dfrac{}{\Sigma; . \vdash^{q+K^{unit}}_{q} () : unit \rightsquigarrow \lambda u.\mathsf{release} - = u \mathsf{\ in\ bind} - = \uparrow^{K^{unit}} \mathsf{in\ bind}\, a = \mathsf{store}() \mathsf{\ in\ ret}(a)} \text{ unit}$$

$E_0 = \lambda u.\mathsf{release} - = u \mathsf{\ in\ bind} - = \uparrow^{K^{unit}} \mathsf{in\ bind}\, a = \mathsf{store}() \mathsf{\ in\ ret}(a)$
$E_1 = \mathsf{release} - = u \mathsf{\ in\ bind} - = \uparrow^{K^{unit}} \mathsf{in\ bind}\, a = \mathsf{store}() \mathsf{\ in\ ret}(a)$
$T_0 = [q + K^{unit}]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q] (\!|unit|\!))$
$T_1 = [q + K^{unit}]\, \mathbf{1}$
$T_2 = \mathbb{M}(q + K^{unit})\, ([q]\, \mathbf{1})$
$T_{2.1} = \mathbb{M}(q)\, ([q]\, \mathbf{1})$
$T_3 = \mathbb{M}\, K^{unit}\, \mathbf{1}$
$T_4 = \mathbb{M}\, 0\, ([q]\, \mathbf{1})$
$T_5 = \mathbb{M}\, q\, ([q]\, \mathbf{1})$
D1:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); . \vdash \mathsf{store}() : T_5} \qquad \dfrac{}{.; .; .; (\!|\Sigma|\!); a : [q]\, \mathbf{1} \vdash \mathsf{ret}(a) : T_4}}{.; .; .; (\!|\Sigma|\!); . \vdash \mathsf{bind}\, a = \mathsf{store}() \mathsf{\ in\ ret}(a) : T_5}$$

D0:

$$\dfrac{}{.; .; .; (\!|\Sigma|\!); . \vdash \uparrow^{K^{unit}} : T_3}$$

D0.0:

$$\dfrac{D0 \qquad D1}{.; .; .; (\!|\Sigma|\!); . \vdash \mathsf{bind} - = \uparrow^{K^{unit}} \mathsf{in\ bind}\, a = \mathsf{store}() \mathsf{\ in\ ret}(a) : T_2} \text{ T-bind}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); u : T_1 \vdash u : T_1} \text{ T-var} \qquad D0.0}{.; .; .; (\!|\Sigma|\!); u : T_1 \vdash E_1 : T_4} \text{ T-release}}{.; .; .; (\!|\Sigma|\!); . \vdash E_0 : T_0} \text{ T-lam}$$

2. base:

$$\dfrac{}{\Sigma; . \vdash^{q+K^{base}}_{q} c : \mathsf{b} \rightsquigarrow \lambda u.\mathsf{release} - = u \mathsf{\ in\ bind} - = \uparrow^{K^{base}} \mathsf{in\ bind}\, a = \mathsf{store}(!c) \mathsf{\ in\ ret}(a)} \text{ base}$$

$E_0 = \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{base}} \text{ in } \mathsf{bind}\, a = \mathsf{store}(!c) \text{ in } \mathsf{ret}(a)$

$E_1 = \mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{base}} \text{ in } \mathsf{bind}\, a = \mathsf{store}(!c) \text{ in } \mathsf{ret}(a)$

$T_0 = [q + K^{base}]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q](\!|\mathsf{b}|\!))$

$T_1 = [q + K^{base}]\, \mathbf{1}$

$T_2 = \mathbb{M}(q + K^{base})\, ([q]\, !\mathsf{b})$

$T_{2.1} = \mathbb{M}(q)\, ([q]\, !\mathsf{b})$

$T_3 = \mathbb{M}\, K^{base}\, (\!|\mathbf{1}|\!)$

$T_4 = \mathbb{M}\, 0\, ([q]\, !\mathsf{b})$

$T_5 = \mathbb{M}\, q\, ([q]\, !\mathsf{b})$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);. \vdash \mathsf{store}(!c) : T_5} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); a : [q]\, !\mathsf{b} \vdash \mathsf{ret}(a) : T_4}}{.;.;.;(\!|\Sigma|\!);. \vdash \mathsf{bind}\, a = \mathsf{store}(!c) \text{ in } \mathsf{ret}(a) : T_{2.1}}$$

D0:

$$\dfrac{}{.;.;.;(\!|\Sigma|\!);. \vdash \uparrow^{K^{base}} : T_3}$$

D0.0:

$$\dfrac{D0 \qquad D1}{.;.;.;(\!|\Sigma|\!);. \vdash \mathsf{bind} - = \uparrow^{K^{base}} \text{ in } \mathsf{bind}\, a = \mathsf{store}(!c) \text{ in } \mathsf{ret}(a) : T_2}\ \text{T-bind}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); u : T_1 \vdash u : T_1}\ \text{T-var} \qquad D0.0}{.;.;.;(\!|\Sigma|\!); u : T_1 \vdash E_1 : T_4}\ \text{T-release}}{.;.;.;(\!|\Sigma|\!);. \vdash E_0 : T_0}\ \text{T-lam}$$

3. var:

$$\dfrac{}{\Sigma; x : \tau \vdash_{q}^{q+K^{var}} x : \tau \rightsquigarrow \lambda u.\, \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{ret}(x)}\ \text{var}$$

$E_0 = \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a)$

$E_1 = \mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a)$

$T_0 = [q + K^{var}]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q](\!|\tau|\!))$

$T_1 = [q + K^{var}]\, \mathbf{1}$

$T_2 = \mathbb{M}\, 0\, ([q + K^{var}](\!|\tau|\!))$

$T_3 = \mathbb{M}\, K^{var}\, (\!|\mathbf{1}|\!)$

$T_4 = \mathbb{M}\, 0\, ([q](\!|\tau|\!))$

$T_5 = \mathbb{M}\, q\, ([q](\!|\tau|\!))$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); x : (\!|\tau|\!) \vdash \mathsf{store}\, x : T_5} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); a : [q]\, (\!|\tau|\!) \vdash \mathsf{ret}(a) : T_4}}{.;.;.;(\!|\Sigma|\!); x : (\!|\tau|\!) \vdash \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a) : T_5}$$

D0:

$$\frac{}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau|\!) \vdash \uparrow^{K^{var}} : T_3}$$

D0.0:

$$\frac{D0 \qquad D1}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau|\!) \vdash \mathsf{bind} - = \uparrow^{K^{var}} \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}\, x\ \mathsf{in}\ \mathsf{ret}(a) : T_2}\ \text{T-bind}$$

Main derivation:

$$\frac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); u : T_1 \vdash u : T_1}\ \text{T-var} \qquad D0.0}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau|\!), u : T_1 \vdash E_1 : T_4}\ \text{T-release}}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau|\!) \vdash E_0 : T_0}\ \text{T-lam}$$

4. app:

$$\frac{\tau_1 \overset{q/q'}{\to} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash^{q+K_1^{app}}_{q'-K_2^{app}} f\ x : \tau_2 \rightsquigarrow \lambda u.E_0}\ \text{app}$$

where

$E_0 = \mathsf{release} - = u\ \mathsf{in}\ \mathsf{bind} - = \uparrow^{K_1^{app}} \mathsf{in}\ \mathsf{bind}\, P = \mathsf{store}()\ \mathsf{in}\ E_1$

$E_1 = \mathsf{bind}\, f_1 = (f\ P\ x)\ \mathsf{in}\ \mathsf{release}\, f_2 = f_1\ \mathsf{in}\ \mathsf{bind} - = \uparrow^{K_2^{app}} \mathsf{in}\ \mathsf{bind}\, f_3 = \mathsf{store}\, f_2\ \mathsf{in}\ \mathsf{ret}\, f_3$

$E_{1.1} = \mathsf{release}\, f_2 = f_1\ \mathsf{in}\ \mathsf{bind} - = \uparrow^{K_2^{app}} \mathsf{in}\ \mathsf{bind}\, f_3 = \mathsf{store}\, f_2\ \mathsf{in}\ \mathsf{ret}\, f_3$

$E_{1.2} = \mathsf{bind} - = \uparrow^{K_2^{app}} \mathsf{in}\ \mathsf{bind}\, f_3 = \mathsf{store}\, f_2\ \mathsf{in}\ \mathsf{ret}\, f_3$

$E_{1.3} = \mathsf{bind}\, f_3 = \mathsf{store}\, f_2\ \mathsf{in}\ \mathsf{ret}\, f_3$

$E_{1.4} = \mathsf{store}\, f_2$

$E_{1.5} = \mathsf{ret}\, f_3$

$E_{0.1} = \mathsf{bind} - = \uparrow^{K_1^{app}} \mathsf{in}\ \mathsf{bind}\, F = f\ \mathsf{in}\ E_1$

$T_0 = [q + K_1^{app}]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q' - K_2^{app}](\!|\tau|\!))$

$T_{0.1} = [q + K_1^{app}]\, \mathbf{1}$

$T_{0.2} = \mathbb{M}\, 0\, ([q' - K_2^{app}](\!|\tau_2|\!))$

$T_1 = \mathbb{M}(q + K_1^{app})\, \mathbf{1}$

$T_{1.2} = \mathbb{M}\, 0\, [q' - K_2^{app}]\, (\!|\tau_2|\!)$

$T_2 = \mathbb{M}(K_1^{app})\, \mathbf{1}$

$T_3 = \mathbb{M}(q)\, (\!|\tau_2|\!)$

$T_4 = \mathbb{M}\, q\, ((\!|\tau_1|\!) \multimap \mathbb{M}\, 0\, [q']\, (\!|\tau_2|\!))$

$T_{4.1} = ((\!|\tau_1|\!) \multimap \mathbb{M}\, 0\, [q']\, (\!|\tau_2|\!))$

$T_{4.2} = \mathbb{M}\, 0\, [q']\, (\!|\tau_2|\!)$

$T_{4.3} = [q']\, (\!|\tau_2|\!)$

$T_{4.4} = \mathbb{M}(\, q' - K_2^{app})[q' - K_2^{app}]\, (\!|\tau_2|\!)$

$T_{4.41} = \mathbb{M}(q')[q' - K_2^{app}] \, (\!|\tau_2|\!)$

$T_{4.5} = [q' - K_2^{app}] \, (\!|\tau_2|\!)$

$T_{4.6} = \mathbb{M}\, 0 \, [q' - K_2^{app}] \, (\!|\tau_2|\!)$

D2.3:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); f_2 : (\!|\tau_2|\!) \vdash E_{1.4} : T_{4.4}} \qquad \dfrac{}{.; .; .; (\!|\Sigma|\!); f_3 : T_{4.5} \vdash E_{1.5} : T_{4.6}}}{.; .; .; (\!|\Sigma|\!); f_2 : (\!|\tau_2|\!), f_3 : T_{4.5} \vdash E_{1.3} : T_{4.4}}$$

D2.2:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); . \vdash \uparrow^{K_2^{app}} : \mathbb{M}\, K_2^{app}\, \mathbf{1}} \qquad D2.3}{.; .; .; (\!|\Sigma|\!); f_2 : (\!|\tau_2|\!) \vdash E_{1.2} : T_{4.41}}$$

D2.1:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); f_1 : T_{4.3} \vdash f_1 : T_{4.3}} \qquad D2.2}{.; .; .; (\!|\Sigma|\!); f_1 : T_{4.3} \vdash E_{1.1} : T_{1.2}}$$

D2:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!), P : [q]\, \mathbf{1} \vdash f \;\, P \;\, x : T_{4.2}} \qquad D2.1}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!), P : [q]\, \mathbf{1} \vdash E_1 : T_{1.2}}$$

D1:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); . \vdash \mathsf{store}() : \mathbb{M}\, q\, [q]\, \mathbf{1}} \qquad D2}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!) \vdash \mathsf{bind}\, P = \mathsf{store}() \;\mathsf{in}\; E_1 : T_{1.2}}$$

D0:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!) \vdash \uparrow^{K_1^{app}} : T_1} \qquad D1}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!) \vdash E_{0.1} : T_{1.2}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); u : T_{0.1} \vdash u : T_{0.1}}\text{T-var} \qquad D0}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!), u : T_{0.1} \vdash E_0 : T_{0.2}}}{.; .; .; (\!|\Sigma|\!); x : (\!|\tau_1|\!) \vdash \lambda u.E_0 : T_0}$$

5. nil:

$$\dfrac{}{\begin{array}{c}\Sigma; \varnothing \vdash_q^{q+K^{nil}} nil : L^{\vec{p}}\tau \rightsquigarrow \\ \lambda u.\mathsf{release} - = u \;\mathsf{in}\; \mathsf{bind} - = \uparrow^{K^{nil}} \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil \rangle\!\rangle \;\mathsf{in}\; \mathsf{ret}(b)\end{array}}\text{nil}$$

$E_0 = \lambda u.\mathsf{release} - = u \;\mathsf{in}\; \mathsf{bind} - = \uparrow^{K^{nil}} \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil \rangle\!\rangle \;\mathsf{in}\; \mathsf{ret}(b)$

$E_1 = \mathsf{release} - = u \;\mathsf{in}\; \mathsf{bind} - = \uparrow^{K^{nil}} \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil \rangle\!\rangle \;\mathsf{in}\; \mathsf{ret}(b)$

$E_2 = \mathsf{bind} - = \uparrow^{K^{nil}} \;\mathsf{in}\; \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil \rangle\!\rangle \;\mathsf{in}\; \mathsf{ret}(b)$

$E_3 = \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil\rangle\!\rangle\ \mathsf{in}\ \mathsf{ret}(b)$

$E_4 = \mathsf{bind}\, b = \mathsf{store}\langle\!\langle a, nil\rangle\!\rangle\ \mathsf{in}\ \mathsf{ret}(b)$

$E_5 = \mathsf{ret}(b)$

$T_0 = [q + K^{nil}]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q]\,\exists n.\phi(\vec{p}, n) \otimes \mathrm{list}[n](\!|\tau|\!))$

$T_1 = [(q + K^{nil})]\,\mathbf{1}$

$T_2 = \mathbb{M}\,0\,([q]\,\exists n.\,[\phi(\vec{p}, n)]\,\mathbf{1} \otimes \mathrm{list}[n](\!|\tau|\!))$

$T_3 = \mathbb{M}(q + K^{nil})\,([q]\,\exists n.\,[\phi(\vec{p}, n)]\,\mathbf{1} \otimes \mathrm{list}[n](\!|\tau|\!))$

$T_4 = \mathbb{M}\,K^{nil}\,\mathbf{1}$

$T_5 = \mathbb{M}(q)\,([q]\,\exists n.\,[\phi(\vec{p}, n)]\,\mathbf{1} \otimes \mathrm{list}[n](\!|\tau|\!))$

$T_{5.1} = ([q]\,\exists n.\,[\phi(\vec{p}, n)]\,\mathbf{1} \otimes \mathrm{list}[n](\!|\tau|\!))$

$T_6 = \mathbb{M}(0)\,([q]\,\exists n.\,[\phi(\vec{p}, n)]\,\mathbf{1} \otimes \mathrm{list}[n](\!|\tau|\!))$

D4:

$$\dfrac{\dfrac{\dfrac{\phi(\vec{p}, 0) = 0}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash a : [0]\,\mathbf{1}} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash nil : \mathrm{list}[0](\!|\tau|\!)}}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash \langle\!\langle a, nil\rangle\!\rangle : T_6[0/n]}}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash \langle\!\langle a, nil\rangle\!\rangle : T_6}$$

D3:

$$\dfrac{}{.;.;.;(\!|\Sigma|\!); b : T_{5.1} \vdash E_5 : T_6}$$

D2:

$$\dfrac{\dfrac{D4}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash \mathsf{store}\langle\!\langle a, nil\rangle\!\rangle : T_5} \qquad D3}{.;.;.;(\!|\Sigma|\!); a : [0]\,\mathbf{1} \vdash E_4 : T_5}$$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); . \vdash \mathsf{store}() : \mathbb{M}\,0\,[0]\,\mathbf{1}} \qquad D2}{.;.;.;(\!|\Sigma|\!); . \vdash E_3 : T_5}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); . \vdash \uparrow^{K^{nil}} : T_4} \qquad D1}{.;.;.;(\!|\Sigma|\!); . \vdash E_2 : T_3}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); u : T_1 \vdash u : T_1} \qquad D0}{.;.;.;(\!|\Sigma|\!); u : T_1 \vdash E_1 : T_2}}{.;.;.;(\!|\Sigma|\!); . \vdash E_0 : T_0}$$

6. cons:

$$\dfrac{\vec{p} = (p_1, \ldots, p_k)}{\begin{array}{c}\Sigma; x_h : \tau, x_t : L^{(\lhd\ \vec{p})}\tau \vdash_q^{q+p_1+K^{cons}} cons(x_h, x_t) : L^{p}\tau \rightsquigarrow \\ \lambda u.\mathsf{release} - = u\ \mathsf{in}\ \mathsf{bind} - = \uparrow^{K^{cons}}\ \mathsf{in}\ E_0\end{array}}\ \text{cons}$$

where

$$E_0 = x_t; x.\,\mathsf{let}\langle\langle x_1, x_2\rangle\rangle = x \ \mathsf{in}\ E_1$$
$$E_1 = \mathsf{release} - = x_1 \ \mathsf{in}\ \mathsf{bind}\, a = \mathsf{store}() \ \mathsf{in}\ \mathsf{store}\langle\langle a, x_h :: x_2\rangle\rangle$$

$$T_0 = [q + p_1 + K^{cons}]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q]\,\exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis)$$
$$T_1 = [q + p_1 + K^{cons}]\,\mathbf{1}$$
$$T_2 = \mathbb{M}\,0\,([q]\,\exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis)$$
$$T_{2.1} = \mathbb{M}(q + p_1)\,([q]\,\exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis)$$
$$T_{2.2} = \mathbb{M}(q + p_1 + \phi(\lhd\vec{p}, s))\,([q]\,\exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis)$$
$$T_{2.3} = \mathbb{M}(q)\,([q]\,\exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis)$$
$$T_{2.4} = \exists n'.\,[\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis$$
$$T_{2.5} = [\phi(\vec{p}, n')]\,\mathbf{1} \otimes L^{n'}\llparenthesis\tau\rrparenthesis$$
$$T_3 = [(p_1 + \phi(\lhd\vec{p}, s))]\,\mathbf{1}$$
$$T_l = \exists s.([\phi(\lhd\vec{p}, s)]\,\mathbf{1} \otimes L^{s}\llparenthesis\tau\rrparenthesis)$$
$$T_{l1} = ([\phi(\lhd\vec{p}, s)]\,\mathbf{1} \otimes L^{s}\llparenthesis\tau\rrparenthesis)$$
$$T_{l2} = [\phi(\lhd\vec{p}, s)]\,\mathbf{1}$$
$$T_{l3} = L^{s}\llparenthesis\tau\rrparenthesis$$

D1.4:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_2 : T_{l3}, a : T_3 \vdash \langle\langle a, x_h :: x_2\rangle\rangle : T_{2.5}[(s+1)/n']} \\ \dfrac{}{s : \mathbb{N} \vdash s + 1 : \mathbb{N}}\end{array}}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_2 : T_{l3}, a : T_3 \vdash \langle\langle a, x_h :: x_2\rangle\rangle : T_{2.4}}}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_2 : T_{l3}, a : T_3 \vdash \mathsf{store}\langle\langle a, x_h :: x_2\rangle\rangle : T_{2.3}}$$

D1.3:

$$\dfrac{\dfrac{}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; . \vdash \mathsf{store}() : \mathbb{M}(p_1 + \phi(\lhd\vec{p}, s))\,[p_1 + \phi(\lhd\vec{p}, s)]\,\mathbf{1}} \quad D1.4}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_2 : T_{l3} \vdash \mathsf{bind}\, a = \mathsf{store}() \ \mathsf{in}\ \mathsf{store}\langle\langle a, x_h :: x_2\rangle\rangle : T_{2.2}}$$

D1.2:

$$\dfrac{\dfrac{}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_1 : T_{l2} \vdash x_1 : T_{l2}} \quad D1.3}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_1 : T_{l2}, x_2 : T_{l3} \vdash E_1 : T_{2.1}}$$

D1.1:

$$\dfrac{\dfrac{}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x : T_{l1} \vdash x : T_{l1}} \quad D1.2}{.; s : \mathbb{N}; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x : T_{l1} \vdash \mathsf{let}\langle\langle x_1, x_2\rangle\rangle = x \ \mathsf{in}\ E_1 : T_{2.1}}$$

D1:

$$\dfrac{\dfrac{}{.; .; .; .; \llparenthesis\Sigma\rrparenthesis; x_t : T_l \vdash x_t : T_l} \quad D1.1}{.; .; .; .; \llparenthesis\Sigma\rrparenthesis; x_h : \llparenthesis\tau\rrparenthesis, x_t : T_l \vdash E_0 : T_{2.1}}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;.;(\!|\Sigma|\!);.\vdash \uparrow^{K^{cons}}} \quad D1}{.;.;.;.;(\!|\Sigma|\!);x_h:(\!|\tau|\!),x_t:T_l\vdash \mathsf{bind}\,-=\uparrow^{K^{cons}}\ \mathsf{in}\ E_0:T_{2.1}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;.;(\!|\Sigma|\!);u:T_1\vdash u:T_1} \quad D0}{.;.;.;.;(\!|\Sigma|\!);x_h:(\!|\tau|\!),x_t:T_l,u:T_1\vdash \mathsf{release}\,-=u\ \mathsf{in}\ \mathsf{bind}\,-=\uparrow^{K^{cons}}\ \mathsf{in}\ E_0:T_2}}{.;.;.;.;(\!|\Sigma|\!);x_h:(\!|\tau|\!),x_t:T_l\vdash \lambda u.\mathsf{release}\,-=u\ \mathsf{in}\ \mathsf{bind}\,-=\uparrow^{K^{cons}}\ \mathsf{in}\ E_0:T_0}$$

7. match:

$$\dfrac{\vec{p}=(p_1,\ldots,p_k) \qquad \begin{array}{c}\Sigma;\Gamma\vdash^{q-K_1^{matN}}_{q'+K_2^{matN}} e_1:\tau'\rightsquigarrow e_{a1}\\ \Sigma;\Gamma,h:\tau,t:L^{(\lhd\,\vec{p})}\tau\vdash^{q+p_1-K_1^{matC}}_{q'+K_2^{matC}} e_2:\tau'\rightsquigarrow e_{a2}\end{array}}{\Sigma;\Gamma,x:L^{p}\tau\vdash^{q}_{q'} \mathsf{match}\ x\ \mathsf{with}\ |nil\mapsto e_1\ |h::t\mapsto e_2:\tau'\rightsquigarrow \qquad \lambda u.E_0}\ \text{match}$$

where

$E_0=\mathsf{release}\,-=u\ \mathsf{in}\ E_{0.1}$

$E_{0.1}=x;a.\,\mathsf{let}\langle\!\langle x_1,x_2\rangle\!\rangle=a\ \mathsf{in}\ E_1$

$E_1=\mathsf{match}\ x_2\ \mathsf{with}\ |nil\mapsto E_2\ |h::l_t\mapsto E_3$

$E_2=\mathsf{bind}\,-=\uparrow^{K_1^{matN}}\ \mathsf{in}\ E_{2.1}$

$E_{2.1}=\mathsf{bind}\,b=\mathsf{store}()\ \mathsf{in}\ E'_2$

$E'_2=\mathsf{bind}\,c=(e_{a1}\ b)\ \mathsf{in}\ E'_{2.1}$

$E'_{2.1}=\mathsf{release}\,d=c\ \mathsf{in}\ E'_{2.2}$

$E'_{2.2}=\mathsf{bind}\,-=\uparrow^{K_2^{matN}}\ \mathsf{in}\ E'_{2.3}$

$E'_{2.3}=\mathsf{release}\,-=x_1\ \mathsf{in}\ \mathsf{store}\,d$

$E_3=\mathsf{bind}\,-=\uparrow^{K_1^{matC}}\ \mathsf{in}\ E_{3.1}$

$E_{3.1}=\mathsf{release}\,-=x_1\ \mathsf{in}\ E_{3.2}$

$E_{3.2}=\mathsf{bind}\,b=\mathsf{store}()\ \mathsf{in}\ E_{3.3}$

$E_{3.3}=\mathsf{bind}\,t=\mathsf{ret}\langle\!\langle b,l_t\rangle\!\rangle\ \mathsf{in}\ E_{3.4}$

$E_{3.4}=\mathsf{bind}\,d=\mathsf{store}()\ \mathsf{in}\ E_{3.5}$

$E_{3.5}=\mathsf{bind}\,f=e_{a2}\ d\ \mathsf{in}\ E_{3.6}$

$E_{3.6}=\mathsf{release}\,g=f\ \mathsf{in}\ E_{3.7}$

$E_{3.7}=\mathsf{bind}\,-=\uparrow^{K_2^{matC}}\ \mathsf{in}\ \mathsf{store}\,g$

$T_0=[q]\,\mathbf{1}\multimap \mathbb{M}\,0\,([q']\,(\!|\tau'|\!))$

$T_1=[q]\,\mathbf{1}$

$T_2=\mathbb{M}\,0\,([q']\,(\!|\tau'|\!))$

$T_{2.0}=\mathbb{M}\,q'\,([q']\,(\!|\tau'|\!))$

$T_{2.1}=\mathbb{M}\,q\,([q']\,(\!|\tau'|\!))$

$T_{2.10} = \mathbb{M}(q - K_1^{matC})\,([q']\,(\!|\tau'|\!))$

$T_{2.11} = \mathbb{M}(q - K_1^{matN})\,([q']\,(\!|\tau'|\!))$

$T_{2.12} = \mathbb{M}(q - K_1^{matN})\,([(q - K^{matN})]\,\mathbf{1})$

$T_{2.13} = ([(q - K_1^{matN})]\,\mathbf{1})$

$T_3 = \mathbb{M}(q - K_1^{matC} + p_1 + \phi(\lhd\vec{p}, i))\,[q']\,(\!|\tau'|\!)$

$T_{3.0} = \mathbb{M}(q - K_1^{matC} + p_1)\,[q']\,(\!|\tau'|\!)$

$T_{3.1} = \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)$

$T_{3.2} = \mathbb{M}(q' + K_2^{matC})\,[q']\,(\!|\tau'|\!)$

$T_{4.0} = \mathbb{M}(\phi(\lhd\vec{p}, i))\,\mathbf{1}$

$T_{4.10} = \mathbb{M}\,0\,T_{4.1}$

$T_{4.1} = \exists s'.([(\phi(\lhd\vec{p}, s'))]\,\mathbf{1} \otimes L^{s'}(\!|\tau|\!))$

$T_{4.11} = ([(\phi(\lhd\vec{p}, i))]\,\mathbf{1} \otimes L^{i}(\!|\tau|\!))$

$T_{4.12} = ([(\phi(\lhd\vec{p}, i))]\,\mathbf{1}$

$T_{4.13} = L^{i}(\!|\tau|\!))$

$T_{4.2} = \mathbb{M}(q - k_1^{matC} + p_1)\,[(q - k_1^{matC} + p_1)]\,\mathbf{1}$

$T_{4.3} = \mathbb{M}\,0\,[(q' + K_2^{matC})]\,(\!|\tau|\!)$

$T_{4.4} = [(q' + K_2^{matC})]\,(\!|\tau|\!)$

$T_b = [\phi(\lhd\vec{p}, s')]\,\mathbf{1}$

$T_c = \exists s'.([\phi(\lhd\vec{p}, s')]\,\mathbf{1} \otimes L^{s'}(\!|\tau|\!))$

$T_d = [q - K_1^{matC} + p_1]\,\mathbf{1}$

$T_f = T_{4.4}$

$T_g = (\!|\tau|\!)$

$T_l = \exists s.([\phi(\vec{p}, s)]\,\mathbf{1} \otimes L^{s}(\!|\tau|\!))$

$T_l' = ([\phi(\vec{p}, s)]\,\mathbf{1} \otimes L^{s}(\!|\tau|\!))$

$T_{l1} = [\phi(\vec{p}, s)]\,\mathbf{1}$

$T_{l2} = L^{s}(\!|\tau|\!)$

$T_{l3} = \exists s'.([\phi(\lhd\vec{p}, s')]\,\mathbf{1} \otimes L^{s'}(\!|\tau|\!))$

$T_{l4} = L^{i}(\!|\tau|\!)$

$T_{ih1} = [q - K_1^{matN}]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q' + K_2^{matN}]\,(\!|\tau'|\!))$

$T_{ih1.1} = \mathbb{M}\,0\,([q' + K_2^{matN}]\,(\!|\tau'|\!))$

$T_{ih1.2} = ([q' + K_2^{matN}]\,(\!|\tau'|\!))$

$T_{ih2} = [q + p_1 - K_1^{matC}]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q' + K_2^{matC}]\,(\!|\tau'|\!))$

$T_{ih2.1} = \mathbb{M}\,0\,([q' + K_2^{matC}]\,(\!|\tau'|\!))$

D3.8:

$$\frac{}{.; s, i; s = i + 1; (\!|\Sigma|\!); g : T_g \vdash \mathsf{store}\, g : \mathbb{M}\, q'\,[q']\,(\!|\tau|\!)}$$

D3.7:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);.\vdash \uparrow^{K_2^{matC}} : \mathbb{M}\,K_2^{matC}\,\mathbf{1}} \quad E_{3.8}}{.;s,i;s=i+1;(\!|\Sigma|\!);g:T_g \vdash E_{3.7}:T_{3.3}}$$

D3.6:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);f:T_f \vdash f:T_f} \quad E_{3.7}}{.;s,i;s=i+1;(\!|\Sigma|\!);f:T_f \vdash E_{3.6}:T_{3.2}}$$

D3.5:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),h:(\!|\tau|\!),t:T_c,d:T_d \vdash e_{a2}\ d:T_{4.3}} \quad E_{3.6}}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),h:(\!|\tau|\!),t:T_c,d:T_d \vdash E_{3.5}:T_{3.2}}$$

D3.4:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);.\vdash \mathsf{store}():T_{4.2}} \quad E_{3.5}}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),h:(\!|\tau|\!),t:T_c \vdash E_{3.4}:T_{3.1}}$$

D3.31:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);l_t:T_{l4},b:T_b \vdash \langle\!\langle b,l_t\rangle\!\rangle:T_{4.11}}}{.;s,i;s=i+1;(\!|\Sigma|\!);l_t:T_{l4},b:T_b \vdash \langle\!\langle b,l_t\rangle\!\rangle:T_{4.1}}$$

D3.3:

$$\dfrac{\dfrac{D3.31}{.;s,i;s=i+1;(\!|\Sigma|\!);l_t:T_{l4},b:T_b \vdash \mathsf{ret}\langle\!\langle b,l_t\rangle\!\rangle:T_{4.10}} \quad D_{3.4}}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),h:(\!|\tau|\!),l_t:T_{l4},b:T_b \vdash E_{3.3}:T_{3.1}}$$

D3.2:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);.\vdash \mathsf{store}():T_{4.0}} \quad D3.3}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!);h:(\!|\tau|\!),l_t:T_{l4} \vdash E_{3.2}:T_3}$$

D3.1:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);x_1:T_{l1} \vdash x_1:T_{l1}} \quad D3.2}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),x_1:T_{l1},h:(\!|\tau|\!),l_t:T_{l4} \vdash E_{3.1}:T_{2.10}}$$

D3:

$$\dfrac{\dfrac{}{.;s,i;s=i+1;(\!|\Sigma|\!);.\vdash \uparrow^{K_1^{matC}} : \mathbb{M}\,K_1^{matC}\,\mathbf{1}} \quad D3.1}{.;s,i;s=i+1;(\!|\Sigma|\!);(\!|\Gamma|\!),x_1:T_{l1},h:(\!|\tau|\!),l_t:T_{l4} \vdash E_3:T_{2.1}}$$

D2.32:

$$\dfrac{\dfrac{}{.;s;s=0;(\!|\Sigma|\!);x_1:T_{l1} \vdash x_1:T_{l1}} \quad \dfrac{}{.;s;s=0;(\!|\Sigma|\!);d:(\!|\tau'|\!) \vdash \mathsf{store}\,d:T_{2.0}}}{.;s;s=0;(\!|\Sigma|\!);x_1:T_{l1},d:(\!|\tau'|\!) \vdash E'_{2.3}:T_{2.0}}$$

D2.31:

$$\dfrac{\dfrac{}{.;s;s=0;(\!|\Sigma|\!);.\vdash \uparrow^{K_2^{matN}} : \mathbb{M}\,K_2^{matN}\,\mathbf{1}} \quad D2.32}{.;s;s=0;(\!|\Sigma|\!);x_1:T_{l1},d:(\!|\tau'|\!) \vdash E'_{2.2}:T_{3.2}}$$

D2.3:

$$\dfrac{\dfrac{}{.; s; s=0; (\!|\Sigma|\!); c : T_{ih1.2} \vdash c : T_{ih1.2}} \quad D2.31}{.; s; s=0; (\!|\Sigma|\!); x_1 : T_{l1}, c : T_{ih1.2} \vdash E'_{2.1} : T_2}$$

D2.22:

$$\dfrac{}{.; s; s=0; (\!|\Sigma|\!); b : T_{2.13} \vdash b : T_{2.13}}$$

D2.21:

$$\dfrac{}{.; s; s=0; (\!|\Sigma|\!); (\!|\Gamma|\!) \vdash e_{a1} : T_{ih1}}$$

D2.2:

$$\dfrac{D2.21 \qquad D2.22}{.; s; s=0; (\!|\Sigma|\!); (\!|\Gamma|\!), b : T_{2.13} \vdash e_{a1}\ b : T_{ih1.1}}$$

D2.20:

$$\dfrac{D2.2 \qquad D2.3}{.; s; s=0; (\!|\Sigma|\!); (\!|\Gamma|\!), x_1 : T_{l1}, b : T_{2.13} \vdash E'_2 : T_2}$$

D2.1:

$$\dfrac{\dfrac{}{.; s; s=0; (\!|\Sigma|\!); . \vdash \mathsf{store}() : T_{2.12}} \quad D2.20}{.; s; s=0; (\!|\Sigma|\!); (\!|\Gamma|\!), x_1 : T_{l1} \vdash E_{2.1} : T_{2.11}}$$

D2:

$$\dfrac{\dfrac{}{.; s; s=0; (\!|\Sigma|\!); . \vdash \uparrow^{K_1^{matN}} : \mathbb{M}\, K_1^{matN}\, \mathbf{1}} \quad D2.1}{.; s; s=0; (\!|\Sigma|\!); (\!|\Gamma|\!), x_1 : T_{l1} \vdash E_2 : T_{2.1}}$$

D1.1:

$$\dfrac{\dfrac{}{.; s; .; (\!|\Sigma|\!); x_2 : T_{l2} \vdash x_2 : T_{l2}} \quad D2 \quad D3}{.; s; .; (\!|\Sigma|\!); (\!|\Gamma|\!), x_1 : T_{l1}, x_2 : T_{l2} \vdash E_1 : T_{2.1}}$$

D1:

$$\dfrac{\dfrac{}{.; s; .; (\!|\Sigma|\!); a : T'_l \vdash a : T'_l} \quad D1.1}{.; s; .; (\!|\Sigma|\!); (\!|\Gamma|\!), a : T'_l \vdash \mathsf{let} \langle\!\langle x_1, x_2 \rangle\!\rangle = a \ \mathsf{in}\ E_1 : T_{2.1}}$$

D0:

$$\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); x : T_l \vdash x : T_l} \quad D1}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), x : T_l \vdash E_{0.1} : T_{2.1}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), x : T_l, u : T_1 \vdash u : T_1} \quad D0}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), x : T_l, u : T_1 \vdash E_0 : T_2}}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), x : T_l \vdash \lambda u.E_0 : T_0}$$

8. Share:

$$\frac{\Sigma;\Gamma, x:\tau_1, y:\tau_2 \vdash^{q}_{q'} e:\tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma;\Gamma, z:\tau \vdash^{q}_{q'} e[z/x, z/y]:\tau' \rightsquigarrow E_0}\ \text{Share-unit}$$

$E_0 = \lambda u.E_1$

$E_1 = \mathsf{bind}\, a = coerce_{\mathbf{1},\mathbf{1},\mathbf{1}}\ z\ \mathsf{in}\ \mathsf{let}\langle\langle x, y\rangle\rangle = a\ \mathsf{in}\ e_a\ u$

$T_0 = [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q']\,(\!|\tau'|\!))$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, x:(\!|\tau_1|\!), y:(\!|\tau_2|\!) \vdash e_a : T_0} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); u:[q]\,\mathbf{1} \vdash u:[q]\,\mathbf{1}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, x:(\!|\tau_1|\!), y:(\!|\tau_2|\!) \vdash e_a\ u : \mathbb{M}\,0\,[q']\,\mathbf{1}}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); a:((\!|\tau_1|\!)\otimes(\!|\tau_2|\!)) \vdash a:((\!|\tau_1|\!)\otimes(\!|\tau_2|\!))} \qquad D1}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, a:((\!|\tau_1|\!)\otimes(\!|\tau_2|\!)) \vdash \mathsf{let}\langle\langle x, y\rangle\rangle = a\ \mathsf{in}\ e_a\ u : [q] \multimap \mathbb{M}\,0\,[q]\,(\!|\tau'|\!)}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{Dc1 \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); z:(\!|\tau|\!) \vdash z:(\!|\tau|\!)}}{.;.;.;(\!|\Sigma|\!); z:(\!|\tau|\!) \vdash coerce_{\mathbf{1},\mathbf{1},\mathbf{1}}\ z : \mathbb{M}\,0\,((\!|\tau_1|\!)\otimes(\!|\tau_2|\!))} \qquad D0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z:(\!|\tau|\!), u:[q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z:(\!|\tau|\!) \vdash \lambda u.E_0 : T_0}$$

$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} : (\!|\mathbf{1}|\!) \multimap \mathbb{M}\,0\,((\!|\mathbf{1}|\!)\otimes(\!|\mathbf{1}|\!))$

$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} \triangleq \lambda u.\ \mathsf{ret}\langle\langle !(), !()\rangle\rangle$

$T_{c0} = (\!|\mathbf{1}|\!) \multimap \mathbb{M}\,0\,((\!|\mathbf{1}|\!)\otimes(\!|\mathbf{1}|\!))$

$T_{c1} = \mathbb{M}\,0\,((\!|\mathbf{1}|\!)\otimes(\!|\mathbf{1}|\!))$

$T_{c2} = (\!|\mathbf{1}|\!)\otimes(\!|\mathbf{1}|\!)$

Dc1:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.;.;.;.;.\vdash \langle\langle !(), !()\rangle\rangle : T_{c2}}}{.;.;.;.; u:(\!|\mathbf{1}|\!) \vdash \langle\langle !(), !()\rangle\rangle : T_{c2}}}{.;.;.;.; u:(\!|\mathbf{1}|\!) \vdash \mathsf{ret}\langle\langle !(), !()\rangle\rangle : T_{c1}}}{.;.;.;.;.\vdash \lambda u.\ \mathsf{ret}\langle\langle !(), !()\rangle\rangle : T_{c0}}$$

$$\frac{\Sigma;\Gamma, x:\tau_1, y:\tau_2 \vdash^{q}_{q'} e:\tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathsf{b}}{\Sigma;\Gamma, z:\tau \vdash^{q}_{q'} e[z/x, z/y]:\tau' \rightsquigarrow E_0}\ \text{Share-base}$$

$E_0 = \lambda u.E_1$

$E_1 = \mathsf{bind}\, a = \mathit{coerce}_{\mathsf{b,b,b}}\ z \text{ in } \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = a \text{ in } e_a\ u$

$T_0 = [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, x:(\!|\tau_1|\!), y:(\!|\tau_2|\!) \vdash e_a : T_0} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); u:[q]\,\mathbf{1} \vdash u:[q]\,\mathbf{1}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, x:(\!|\tau_1|\!), y:(\!|\tau_2|\!) \vdash e_a\ u : \mathbb{M}\,0\,[q']\,\mathbf{1}}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); a:((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash a:((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D1}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u:[q]\,\mathbf{1}, a:((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = a \text{ in } e_a\ u : [q] \multimap \mathbb{M}\,0\,[q]\,(\!|\tau'|\!)}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{Dc1 \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); z:(\!|\tau|\!) \vdash z:(\!|\tau|\!)}}{.;.;.;(\!|\Sigma|\!); z:(\!|\tau|\!) \vdash \mathit{coerce}_{\mathsf{b,b,b}}\ z : \mathbb{M}\,0\,((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z:(\!|\tau|\!), u:[q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z:(\!|\tau|\!) \vdash \lambda u.E_0 : T_0}$$

$\mathit{coerce}_{\mathsf{b,b,b}} : (\!|\mathsf{b}|\!) \multimap \mathbb{M}\,0\,((\!|\mathsf{b}|\!) \otimes (\!|\mathsf{b}|\!))$

$\mathit{coerce}_{\mathsf{b,b,b}} \triangleq \lambda u.\, \mathsf{let}\,!\,u' = u \text{ in } \mathsf{ret}\langle\!\langle !u', !u'\rangle\!\rangle$

$T_{c0} = (\!|\mathsf{b}|\!) \multimap \mathbb{M}\,0\,((\!|\mathsf{b}|\!) \otimes (\!|\mathsf{b}|\!))$

$T_{c1} = \mathbb{M}\,0\,((\!|\mathsf{b}|\!) \otimes (\!|\mathsf{b}|\!))$

$T_{c2} = (\!|\mathsf{b}|\!) \otimes (\!|\mathsf{b}|\!)$

Dc2:

$$\dfrac{\dfrac{}{.;.;.;u':\mathsf{b};. \vdash \langle\!\langle !u', !u'\rangle\!\rangle : T_{c2}}}{.;.;.;u':\mathsf{b};. \vdash \mathsf{ret}\langle\!\langle !u', !u'\rangle\!\rangle : T_{c1}}$$

Dc1:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;.;u :!\mathsf{b} \vdash u :!\mathsf{b}} \qquad Dc2}{.;.;.;.;u:(\!|\mathsf{b}|\!) \vdash \mathsf{let}\,!\,u' = u \text{ in } \mathsf{ret}\langle\!\langle !u', !u'\rangle\!\rangle : T_{c1}}}{.;.;.;.;. \vdash \lambda u.\, \mathsf{let}\,!\,u' = u \text{ in } \mathsf{ret}\langle\!\langle !u', !u'\rangle\!\rangle : T_{c0}}$$

$$\dfrac{\begin{array}{c}\Sigma;\Gamma, x:\tau_1, y:\tau_2 \vdash^{q}_{q'} e:\tau' \rightsquigarrow e_a \\ \tau = L^{\vec{p}}\tau'' \qquad \tau_1 = L^{\vec{p_1}}\tau_1'' \qquad \tau_2 = L^{\vec{p_2}}\tau_2'' \qquad \tau'' = \tau_1'' \curlyvee \tau_2'' \qquad \vec{p} = \vec{p_1} + \vec{p_2}\end{array}}{\Sigma;\Gamma, z:\tau \vdash^{q}_{q'} e[z/x, z/y]:\tau' \rightsquigarrow E_0}\ \text{Share-list}$$

$E_0 = \lambda u.E_1$

$E_1 = \mathsf{bind}\, a = coerce_{\tau,\tau_1,\tau_2}\ z \text{ in } \mathsf{let}\langle\langle x, y\rangle\rangle = a \text{ in } e_a\ u$

$T_0 = [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q']\,(\!|\tau'|\!))$

D1:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u : [q]\,\mathbf{1}, x : (\!|\tau_1|\!), y : (\!|\tau_2|\!) \vdash e_a : T_0} \qquad \dfrac{}{.;.;.;(\!|\Sigma|\!); u : [q]\,\mathbf{1} \vdash u : [q]\,\mathbf{1}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u : [q]\,\mathbf{1}, x : (\!|\tau_1|\!), y : (\!|\tau_2|\!) \vdash e_a\ u : \mathbb{M}\,0\,[q']\,\mathbf{1}}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D1}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), u : [q]\,\mathbf{1}, a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash \mathsf{let}\langle\langle x, y\rangle\rangle = a \text{ in } e_a\ u : [q] \multimap \mathbb{M}\,0\,[q]\,(\!|\tau'|\!)}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); z : (\!|\tau|\!) \vdash coerce_{\tau,\tau_1,\tau_2}\ z : \mathbb{M}\,0\,((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z : (\!|\tau|\!), u : [q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!), z : (\!|\tau|\!) \vdash \lambda u.E_0 : T_0}$$

$coerce_{L^{\vec{p}}\tau, L^{\vec{p_1}}\tau_1, L^{\vec{p_2}}\tau_2} :!((\!|\tau|\!) \multimap \mathbb{M}\,0\,(\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \multimap (\!|L^{\vec{p}}\tau|\!) \multimap \mathbb{M}\,0\,(\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!)$
$coerce_{L^{\vec{p}}\tau, L^{\vec{p_1}}\tau_1, L^{\vec{p_2}}\tau_2} \triangleq \mathsf{fix} f.\lambda\_g.\lambda e.\ \mathsf{let}\,!\,g' = g \text{ in } e; x.\ \mathsf{let}\langle\langle p, l\rangle\rangle = x \text{ in } E_0$

where

$E_0 \triangleq \mathsf{release} - = p \text{ in } E_1$

$E_1 \triangleq \mathsf{match}\ l \text{ with } |nil \mapsto E_{2.1}\ |h :: t \mapsto E_3$

$E_{2.1} \triangleq \mathsf{bind}\, z_1 = \mathsf{store}() \text{ in } E_{2.2}$

$E_{2.2} \triangleq \mathsf{bind}\, z_2 = \mathsf{store}() \text{ in } E_{2.3}$

$E_{2.3} \triangleq \mathsf{ret}\langle\langle\langle\langle z_1, nil\rangle\rangle, \langle\langle z_2, nil\rangle\rangle\rangle\rangle$

$E_3 \triangleq \mathsf{bind}\, H = g'\ h \text{ in } E_{3.1}$

$E_{3.1} \triangleq \mathsf{bind}\, o_t = () \text{ in } E_{3.2}$

$E_{3.2} \triangleq \mathsf{bind}\, T = f\ g\ \langle\langle o_t, t\rangle\rangle \text{ in } E_4$

$E_4 \triangleq \mathsf{let}\langle\langle H_1, H_2\rangle\rangle = H \text{ in } E_5$

$E_5 \triangleq \mathsf{let}\langle\!\langle T_1, T_2\rangle\!\rangle = T \text{ in } E_6$

$E_6 \triangleq T_1; tp_1.\, \mathsf{let}\langle\!\langle p_1', l_1'\rangle\!\rangle = tp_1 \text{ in } E_{7.1}$

$E_{7.1} \triangleq T_2; tp_2.\, \mathsf{let}\langle\!\langle p_2', l_2'\rangle\!\rangle = tp_2 \text{ in } E_{7.2}$

$E_{7.2} \triangleq \mathsf{release} - = p_1' \text{ in } E_{7.3}$

$E_{7.3} \triangleq \mathsf{release} - = p_2' \text{ in } E_{7.4}$

$E_{7.4} \triangleq \mathsf{bind}\, o_1 = \mathsf{store}() \text{ in } E_{7.5}$

$E_{7.5} \triangleq \mathsf{bind}\, o_2 = \mathsf{store}() \text{ in } E_8$

$E_8 \triangleq \mathsf{ret}\langle\!\langle\langle\!\langle o_1, H_1 :: T_1\rangle\!\rangle, \langle\!\langle o_2, H_2 :: T_2\rangle\!\rangle\rangle\!\rangle$

$T_0 = !((\!|\tau|\!) \multimap \mathbb{M}\, 0\, ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))) \multimap (\!|L^{\vec{p}}\tau|\!) \multimap \mathbb{M}\, 0\, ((\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!))$
$T_1 = !((\!|\tau|\!) \multimap \mathbb{M}\, 0\, ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)))$
$T_1' = ((\!|\tau|\!) \multimap \mathbb{M}\, 0\, ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)))$
$T_{1.0} = \exists s.([\phi(\vec{p}, s)]\, \mathbf{1} \otimes L^s(\!|\tau|\!))$
$T_{1.1} = ([\phi(\vec{p}, s)]\, \mathbf{1} \otimes L^s(\!|\tau|\!))$
$T_{1.2} = [\phi(\vec{p}, s)]\, \mathbf{1}$
$T_{1.3} = L^s(\!|\tau|\!)$
$T_2 = (\!|L^{\vec{p}}\tau|\!) \multimap \mathbb{M}\, 0\, ((\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!))$
$T_3 = \mathbb{M}\, 0\, ((\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!))$
$T_{3.1} = \mathbb{M}(\phi(\vec{p}, s))\, ((\!|L^{\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\vec{p_2}}\tau_2|\!))$
$T_{3.11} = \mathbb{M}(\phi(\lhd\vec{p}, s-1))\, ([(\phi(\lhd\vec{p}, s-1))]\, \mathbf{1})$
$T_{3.12} = [(\phi(\lhd\vec{p}, s-1))]\, \mathbf{1}$
$T_4 = \mathbb{M}\, 0\, ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))$
$T_{4.1} = ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))$
$T_5 = \mathbb{M}\, 0\, ((\!|L^{\lhd\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\lhd\vec{p_2}}\tau_2|\!))$
$T_{5.1} = ((\!|L^{\lhd\vec{p_1}}\tau_1|\!) \otimes (\!|L^{\lhd\vec{p_2}}\tau_2|\!))$
$T_{5.2} = (\!|L^{\lhd\vec{p_1}}\tau_1|\!) = \exists s_1'.([\phi(\lhd\vec{p_1}, s_1')]\, \mathbf{1} \otimes L^{s_1'}(\!|\tau_1|\!))$
$T_{5.21} = ([\phi(\lhd\vec{p_1}, s_1')]\, \mathbf{1} \otimes L^{s_1'}(\!|\tau_1|\!))$
$T_{5.22} = [\phi(\lhd\vec{p_1}, s_1')]\, \mathbf{1}$
$T_{5.23} = L^{s_1'}(\!|\tau_1|\!)$
$T_{5.3} = (\!|L^{\lhd\vec{p_2}}\tau_2|\!) = \exists s_2'.([\phi(\lhd\vec{p_2}, s_2')]\, \mathbf{1} \otimes L^{s_2'}(\!|\tau_2|\!))$
$T_{5.31} = ([\phi(\lhd\vec{p_2}, s_2')]\, \mathbf{1} \otimes L^{s_2'}(\!|\tau_2|\!))$
$T_{5.32} = [\phi(\lhd\vec{p_2}, s_2')]\, \mathbf{1}$

$T_{5.33} = L^{s'_2}(\!|\tau_2|\!)$

$P_1 = \vec{p_1} \downarrow_1 + \phi(\lhd \vec{p_1}, s'_1)$

$P_2 = \vec{p_2} \downarrow_1 + \phi(\lhd \vec{p_2}, s'_2)$

$T_6 = \mathbb{M}\, P_1\, ([P_1]\, \mathbf{1})$

$T_{6.1} = [P_1]\, \mathbf{1}$

$T_7 = \mathbb{M}\, P_2\, ([P_2]\, \mathbf{1})$

$T_{7.1} = [P_2]\, \mathbf{1}$

$T_{8.0} = \mathbb{M}(\vec{p} \downarrow_1)\, ((\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!))$

$T_{8.1} = \mathbb{M}(\vec{p} \downarrow_1 + P_1)\, ((\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!))$

$T_{8.2} = \mathbb{M}(\vec{p} \downarrow_1 + P_1 + P_2)\, ((\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!))$

$T_{8.3} = \mathbb{M}(\vec{p_2} \downarrow_1 + P_2)\, ((\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!))$

$T_{8.4} = \mathbb{M}\, 0\, ((\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!))$

$T_{8.41} = (\!|L^{\vec{p_1}} \tau_1|\!) \otimes (\!|L^{\vec{p_2}} \tau_2|\!)$

$T_{8.5} = (\!|L^{\vec{p_1}} \tau_1|\!)$

$T_{8.51} = \exists s_1.([\phi(\vec{p_1}, s_1)]\, \mathbf{1} \otimes L[s_1](\!|\tau_1|\!))$

$T_{8.52} = ([\phi(\vec{p_1}, s'_1)]\, \mathbf{1} \otimes L^{s'_1}(\!|\tau_1|\!))$

$T_{8.6} = (\!|L^{\vec{p_2}} \tau_2|\!)$

$T_{8.61} = \exists s_2.([\phi(\vec{p_2}, s_2)]\, \mathbf{1} \otimes L[s_2](\!|\tau_2|\!))$

$T_{8.62} = ([\phi(\vec{p_2}, s'_2)]\, \mathbf{1} \otimes L^{s'_2}(\!|\tau_2|\!))$

D1.82:

$$\dfrac{\dfrac{}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_2 : (\!|\tau_2|\!), l'_2 : T_{5.33}, o_2 : T_{7.1} \vdash \langle\!\langle o_2, H_2 :: l'_2 \rangle\!\rangle : T_{8.62}}}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_2 : (\!|\tau_2|\!), l'_2 : T_{5.33}, o_2 : T_{7.1} \vdash \langle\!\langle o_2, H_2 :: l'_2 \rangle\!\rangle : T_{8.61}}$$

D1.81:

$$\dfrac{\dfrac{}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), l'_1 : T_{5.23}, o_1 : T_{6.1} \vdash \langle\!\langle o_1, H_1 :: l'_1 \rangle\!\rangle : T_{8.52}}}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), l'_1 : T_{5.23}, o_1 : T_{6.1} \vdash \langle\!\langle o_1, H_1 :: l'_1 \rangle\!\rangle : T_{8.51}}$$

D1.8:

$$\dfrac{\dfrac{\dfrac{D1.81 \qquad D1.182}{\begin{array}{c}.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash \\ \langle\!\langle \langle\!\langle o_1, H_1 :: l'_1 \rangle\!\rangle, \langle\!\langle o_2, H_2 :: l'_2 \rangle\!\rangle \rangle\!\rangle : T_{8.41}\end{array}}}{\begin{array}{c}.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash \\ \mathsf{ret} \langle\!\langle \langle\!\langle o_1, H_1 :: l'_1 \rangle\!\rangle, \langle\!\langle o_2, H_2 :: l'_2 \rangle\!\rangle \rangle\!\rangle : T_{8.4}\end{array}}}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash E_8 : T_{8.4}}$$

D1.75:

$$\dfrac{\dfrac{\dfrac{}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; . \vdash \mathsf{store}() : T_7} \qquad D1.8}{\begin{array}{c}.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1} \vdash \\ \mathsf{bind}\, o_2 = \mathsf{store}()\ \mathsf{in}\ E_8 : T_{8.3}\end{array}}}{.; s'_2, s'_1, s; .; g' : T'_1, f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1} \vdash E_{7.5} : T_{8.3}}$$

D1.74:

$$\dfrac{\dfrac{\dfrac{}{.; s_2', s_1', s; .; g' : T_1', f : T_0; . \vdash \mathsf{store}() : T_6}\ D1.75}{.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l_1' : T_{5.23}, l_2' : T_{5.33} \vdash \mathsf{bind}\, o_1 = \mathsf{store}() \ \mathsf{in}\ E_{7.5} : T_{8.2}}}{.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l_1' : T_{5.23}, l_2' : T_{5.33} \vdash E_{7.4} : T_{8.2}}$$

D1.73:

$$\dfrac{\dfrac{\dfrac{}{.; s_2', s_1', s; .; g' : T_1', f : T_0; p_2' : T_{5.32} \vdash p_2' : T_{5.32}}\ D1.74}{\begin{array}{c}.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l_1' : T_{5.23}, p_2' : T_{5.32}, l_2' : T_{5.33} \vdash \\ \mathsf{release} - = p_2' \ \mathsf{in}\ E_{7.4} : T_{8.1}\end{array}}}{.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), l_1' : T_{5.23}, p_2' : T_{5.32}, l_2' : T_{5.33} \vdash E_{7.3} : T_{8.1}}$$

D1.72:

$$\dfrac{\dfrac{\dfrac{}{.; s_2', s_1', s; .; g' : T_1', f : T_0; p_1' : T_{5.22} \vdash p_1' : T_{5.22}}\ D1.73}{\begin{array}{c}.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), p_1' :, l_1' :, p_2' : T_{5.32}, l_2' : T_{5.33} \vdash \\ \mathsf{release} - = p_1' \ \mathsf{in}\ E_{7.3} : T_{8.0}\end{array}}}{.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), p_1' :, l_1' :, p_2' : T_{5.32}, l_2' : T_{5.33} \vdash E_{7.2} : T_{8.0}}$$

D1.711:

$$\dfrac{\dfrac{}{.; s_2', s_1', s; .; g' : T_1', f : T_0; tp_2 : T_{5.31} \vdash tp_2 : T_{5.31}}\ D1.72}{\begin{array}{c}.; s_2', s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), p_1' : T_{5.22}, l_1' : T_{5.23}, tp_2 : T_{5.31} \vdash \\ \mathsf{let}\langle\!\langle p_2', l_2' \rangle\!\rangle = tp_2 \ \mathsf{in}\ E_{7.2} : T_{8.0}\end{array}}$$

D1.71:

$$\dfrac{\dfrac{\dfrac{}{.; s_1', s; .; g' : T_1', f : T_0; T_2 : T_{5.3} \vdash T_2 : T_{5.3}}\ D1.711}{\begin{array}{c}.; s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T_2 : T_{5.3}, p_1' : T_{5.22}, l_1' : T_{5.23} \vdash \\ T_2; tp_2.\, \mathsf{let}\langle\!\langle p_2', l_2' \rangle\!\rangle = tp_2 \ \mathsf{in}\ E_{7.2} : T_{8.0}\end{array}}}{.; s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T_2 : T_{5.3}, p_1' : T_{5.22}, l_1' : T_{5.23} \vdash E_7 : T_{8.0}}$$

D1.61:

$$\dfrac{\dfrac{}{.; s_1', s; .; g' : T_1', f : T_0; tp_1 : T_{5.21} \vdash tp_1 : T_{5.21}}\ D1.71}{.; s_1', s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T_2 : T_{5.3}, tp_1 : T_{5.21} \vdash \mathsf{let}\langle\!\langle p_1', l_1' \rangle\!\rangle = tp_1 \ \mathsf{in}\ E_7 : T_{8.0}}$$

D1.6:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; T_1 : T_{5.2} \vdash T_1 : T_{5.2}}}{\begin{array}{c}.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T_1 : T_{5.2}, T_1 : T_{5.3} \vdash \\ T_1; tp_1.\, \mathsf{let}\langle\!\langle p_1', l_1' \rangle\!\rangle = tp_1 \ \mathsf{in}\ E_7 : T_{8.0}\end{array}}\ D1.61}{.; s; .; g' : T_1', f : T_0; H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T_1 : T_{5.2}, T_1 : T_{5.3} \vdash E_6 : T_{8.0}}$$

D1.5:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; T : T_{5.1} \vdash T : T_{5.1}} \quad D1.6}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T : T_{5.1} \vdash \mathsf{let}\langle\!\langle T_1, T_2 \rangle\!\rangle = T \ \mathsf{in}\ E_6 : T_{8.0}}}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H_1 : (\!|\tau_1|\!), H_2 : (\!|\tau_2|\!), T : T_{5.1} \vdash E_5 : T_{8.0}}$$

D1.4:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; H : T_{4.1} \vdash H : T_{4.1}} \quad D1.5}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H : T_{4.1}, T : T_{5.1} \vdash \mathsf{let}\langle\!\langle H_1, H_2 \rangle\!\rangle = H \ \mathsf{in}\ E_5 : T_{8.0}}}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H : T_{4.1}, T : T_{5.1} \vdash E_4 : T_{8.0}}$$

D1.3:

$$\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; t : L^{s-1}(\!|\tau|\!), o_t : T_{3.12} \vdash f\ \langle\!\langle o_t, t \rangle\!\rangle : T_5} \quad D1.4}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, H : T_{4.1}, t : L^{s-1}(\!|\tau|\!), o_t : T_{3.12} \vdash \mathsf{bind}\, T = f\ \langle\!\langle o_t, t \rangle\!\rangle \ \mathsf{in}\ E_4 : T_{8.0}}$$

D1.21:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, h : (\!|\tau|\!), t : L^{s-1}(\!|\tau|\!) \vdash \mathsf{store}() : T_{3.11}} \quad D1.3}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, h : (\!|\tau|\!), t : L^{s-1}(\!|\tau|\!) \vdash \mathsf{bind}\, o_t = \mathsf{store}() \ \mathsf{in}\ E_{3.2} : T_{3.1}}}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, h : (\!|\tau|\!), t : L^{s-1}(\!|\tau|\!) \vdash E_{3.1} : T_{3.1}}$$

D1.2:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; h : (\!|\tau|\!) \vdash g'\ h : T_4} \quad D1.3}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, h : (\!|\tau|\!), t : L^{s-1}(\!|\tau|\!) \vdash \mathsf{bind}\, H = g'\ h \ \mathsf{in}\ E_{3.1} : T_{3.1}}}{.; s; .; g' : T_1', f : T_0; p : T_{1.2}, h : (\!|\tau|\!), t : L^{s-1}(\!|\tau|\!) \vdash E_3 : T_{3.1}}$$

D1.14:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; z_2 : [0]\,\mathbf{1} \vdash z_2 : [0]\,\mathbf{1}} \quad \dfrac{}{.; s; .; g' : T_1', f : T_0; z_2 : [0]\,\mathbf{1} \vdash nil : L^0(\!|\tau_2|\!)}}{.; s; .; g' : T_1', f : T_0; z_2 : [0]\,\mathbf{1} \vdash \langle\!\langle z_2, nil \rangle\!\rangle : ([0]\,\mathbf{1} \otimes L^0(\!|\tau_2|\!))}}{.; s; .; g' : T_1', f : T_0; z_2 : [0]\,\mathbf{1} \vdash \langle\!\langle z_2, nil \rangle\!\rangle : \exists s'.([s']\,\mathbf{1} \otimes L^{s'}(\!|\tau_2|\!))}$$

D1.13:

$$\dfrac{\dfrac{\dfrac{}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1} \vdash z_1 : [0]\,\mathbf{1}} \quad \dfrac{}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1} \vdash nil : L^0(\!|\tau_1|\!)}}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1} \vdash \langle\!\langle z_1, nil \rangle\!\rangle : ([0]\,\mathbf{1} \otimes L^0(\!|\tau_1|\!))}}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1} \vdash \langle\!\langle z_1, nil \rangle\!\rangle : \exists s'.([s']\,\mathbf{1} \otimes L^{s'}(\!|\tau_1|\!))}$$

D1.12:

$$\dfrac{\dfrac{\dfrac{D1.13 \quad D1.14}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1}, z_2 : [0]\,\mathbf{1} \vdash \langle\!\langle \langle\!\langle z_1, nil \rangle\!\rangle, \langle\!\langle z_2, nil \rangle\!\rangle \rangle\!\rangle : T_{3.2}}}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1}, z_2 : [0]\,\mathbf{1} \vdash \mathsf{ret}\langle\!\langle \langle\!\langle z_1, nil \rangle\!\rangle, \langle\!\langle z_2, nil \rangle\!\rangle \rangle\!\rangle : T_{3.1}}}{.; s; .; g' : T_1', f : T_0; z_1 : [0]\,\mathbf{1}, z_2 : [0]\,\mathbf{1} \vdash E_{2.3} : T_{3.1}}$$

D1.11:

$$\dfrac{\dfrac{\dfrac{}{.;s;.;g':T_1',f:T_0;.\vdash \mathsf{store}():\mathbb{M}\,0\,[0]\,\mathbf{1}} \qquad D1.12}{.;s;.;g':T_1',f:T_0;z_1:[0]\,\mathbf{1}\vdash \mathsf{bind}\,z_2=\mathsf{store}()\ \mathsf{in}\ E_{2.3}:T_{3.1}}}{.;s;.;g':T_1',f:T_0;z_1:[0]\,\mathbf{1}\vdash E_{2.2}:T_{3.1}}$$

D1.10:

$$\dfrac{\dfrac{\dfrac{}{.;s;.;g':T_1',f:T_0;.\vdash \mathsf{store}():\mathbb{M}\,0\,[0]\,\mathbf{1}} \qquad D1.11}{.;s;.;g':T_1',f:T_0;.\vdash \mathsf{bind}\,z_1=\mathsf{store}()\ \mathsf{in}\ E_{2.2}:T_{3.1}}}{.;s;.;g':T_1',f:T_0;.\vdash E_{2.1}:T_{3.1}}$$

D1:

$$\dfrac{\dfrac{}{.;s;.;g':T_1',f:T_0;l:T_{1.3}\vdash l:T_{1.3}} \qquad D1.10 \qquad D1.2}{.;s;.;g':T_1',f:T_0;p:T_{1.2},l:T_{1.3}\vdash \mathsf{match}\ l\ \mathsf{with}\ |nil\mapsto E_2\ |h::t\mapsto E_3:T_{3.1}}$$

D0.3:

$$\dfrac{\dfrac{\dfrac{}{.;s;.;g':T_1',f:T_0;p:T_{1.2}\vdash p:T_{1.2}} \qquad D1}{.;s;.;g':T_1',f:T_0;p:T_{1.2},l:T_{1.3}\vdash \mathsf{release}-=p\ \mathsf{in}\ E_1:T_3}}{.;s;.;g':T_1',f:T_0;p:T_{1.2},l:T_{1.3}\vdash E_0:T_3}$$

D0.2:

$$\dfrac{\dfrac{}{.;s;.;g':T_1',f:T_0;x:T_{1.1}\vdash x:T_{1.1}} \qquad D0.3}{.;s;.;g':T_1',f:T_0;x:T_{1.1}\vdash \mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_3}$$

D0.1:

$$\dfrac{\dfrac{}{.;.;.;g':T_1',f:T_0;e:(\!|L^p\tau|\!)\vdash e:(\!|L^p\tau|\!)} \qquad D0.2}{.;.;.;g':T_1',f:T_0;e:(\!|L^p\tau|\!)\vdash e;x.\,\mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_3}$$

D0:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{.;.;.;f:T_0;g:T_1\vdash g:T_1} \qquad D1.1}{.;.;.;f:T_0;g:T_1,e:(\!|L^p\tau|\!)\vdash \mathsf{let}\,!\,g'=g\ \mathsf{in}\ e;x.\,\mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_3}}{.;.;.;f:T_0;g:T_1\vdash \lambda e.\,\mathsf{let}\,!\,g'=g\ \mathsf{in}\ e;x.\,\mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_2}}{.;.;.;f:T_0;.\vdash \lambda\_g.\lambda e.\,\mathsf{let}\,!\,g'=g\ \mathsf{in}\ e;x.\,\mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_0}}{.;.;.;.;.\vdash \mathsf{fix}f.\lambda\_g.\lambda e.\,\mathsf{let}\,!\,g'=g\ \mathsf{in}\ e;x.\,\mathsf{let}\langle\!\langle p,l\rangle\!\rangle=x\ \mathsf{in}\ E_0:T_0}$$

$$\dfrac{\begin{array}{c}\Sigma;\Gamma,x:\tau_1,y:\tau_2\vdash^q_{q'} e:\tau'\rightsquigarrow e_a\\ \tau=\tau_1\curlyvee\tau_2 \qquad \tau=(\tau_a,\tau_b) \qquad \tau_1=(\tau_a',\tau_b') \qquad \tau_2=(\tau_a'',\tau_b'')\end{array}}{\Sigma;\Gamma,z:\tau\vdash^q_{q'} e[z/x,z/y]:\tau'\rightsquigarrow E_0}\ \text{Share-pair}$$

$E_0=\lambda u.E_1$

$E_1=\mathsf{bind}\,a=coerce_{(\tau_a,\tau_b),(\tau_a',\tau_b'),(\tau_a'',\tau_b'')}\ z\ \mathsf{in}\ \mathsf{let}\langle\!\langle x,y\rangle\!\rangle=a\ \mathsf{in}\ e_a\ u$

$T_0=[q]\,\mathbf{1}\multimap\mathbb{M}\,0\,([q']\,(\!|\tau'|\!))$

D1:

$$\dfrac{\overline{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), u : [q]\,\mathbf{1}, x : (\!|\tau_1|\!), y : (\!|\tau_2|\!) \vdash e_a : T_0} \qquad \overline{.; .; .; (\!|\Sigma|\!); u : [q]\,\mathbf{1} \vdash u : [q]\,\mathbf{1}}}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), u : [q]\,\mathbf{1}, x : (\!|\tau_1|\!), y : (\!|\tau_2|\!) \vdash e_a\ u : \mathbb{M}\,0\,[q']\,\mathbf{1}}$$

D0:

$$\dfrac{\overline{.; .; .; (\!|\Sigma|\!); a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D1}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), u : [q]\,\mathbf{1}, a : ((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) \vdash \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = a\ \mathsf{in}\ e_a\ u : [q] \multimap \mathbb{M}\,0\,[q]\,(\!|\tau'|\!)}$$

Main derivation:

$$\dfrac{\dfrac{\overline{.; .; .; (\!|\Sigma|\!); z : (\!|\tau|\!) \vdash coerce_{(\tau_a,\tau_b),(\tau'_a,\tau'_b),(\tau''_a,\tau''_b)}\ z : \mathbb{M}\,0\,((\!|\tau_1|\!) \otimes (\!|\tau_2|\!))} \qquad D0}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), z : (\!|\tau|\!), u : [q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,[q']\,(\!|\tau'|\!)}}{.; .; .; (\!|\Sigma|\!); (\!|\Gamma|\!), z : (\!|\tau|\!) \vdash \lambda u.E_0 : T_0}$$

$coerce_{(\tau_a,\tau_b),(\tau'_a,\tau_b),(\tau''_a,\tau''_b)} :!((\!|\tau_a|\!) \multimap \mathbb{M}\,0\,(\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!)) \multimap!((\!|\tau_b|\!) \multimap \mathbb{M}\,0\,(\!|\tau'_b|\!) \otimes (\!|\tau''_b|\!)) \multimap (\!|(\tau_a,\tau_b)|\!) \multimap \mathbb{M}\,0\,(\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!)$

$coerce_{(\tau_a,\tau_b),(\tau'_a,\tau'_b),(\tau''_a,\tau''_b)} \triangleq \lambda\_g_1.\lambda\_g_2.\lambda p.\,\mathsf{let}\,!\langle\!\langle p_1, p_2\rangle\!\rangle = p\ \mathsf{in}\ E_0$

where

$E_0 \triangleq \mathsf{let}\,!\,g'_1 = g_1\ \mathsf{in}\ E_1$

$E_1 \triangleq \mathsf{let}\,!\,g'_2 = g_2\ \mathsf{in}\ E_2$

$E_2 \triangleq \mathsf{bind}\,P'_1 = g'_1 p_1\ \mathsf{in}\ E_3$

$E_3 \triangleq \mathsf{bind}\,P'_2 = g'_2 p_2\ \mathsf{in}\ E_4$

$E_4 \triangleq \mathsf{let}\,!\langle\!\langle p'_{11}, p'_{12}\rangle\!\rangle = P'_1\ \mathsf{in}\ E_5$

$E_5 \triangleq \mathsf{let}\,!\langle\!\langle p'_{21}, p'_{22}\rangle\!\rangle = P'_2\ \mathsf{in}\ E_6$

$E_6 \triangleq \mathsf{ret}\langle\!\langle p'_{11}, p'_{21}\rangle\!\rangle, \langle\!\langle p'_{12}, p'_{22}\rangle\!\rangle$

$T_0 =!((\!|\tau_a|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!))) \multimap!((\!|\tau_b|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_b|\!) \otimes (\!|\tau''_b|\!))) \multimap$
$(\!|(\tau_a,\tau_b)|\!) \multimap \mathbb{M}\,0\,((\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!))$

$T_{0.31} =!((\!|\tau_a|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!)))$

$T_{0.32} = ((\!|\tau_a|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!)))$

$T_{0.4} =!((\!|\tau_b|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_b|\!) \otimes (\!|\tau''_b|\!))) \multimap (\!|(\tau_a,\tau_b)|\!) \multimap \mathbb{M}\,0\,((\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!))$

$T_{0.41} =!((\!|\tau_b|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_b|\!) \otimes (\!|\tau''_b|\!)))$

$T_{0.42} = ((\!|\tau_b|\!) \multimap \mathbb{M}\,0\,((\!|\tau'_b|\!) \otimes (\!|\tau''_b|\!)))$

$T_{0.5} = (\!|(\tau_a,\tau_b)|\!) \multimap \mathbb{M}\,0\,((\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!))$

$T_{0.51} = (\!|(\tau_a,\tau_b)|\!)$

$T_{0.6} = \mathbb{M}\,0\,((\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!))$

$T_{0.61} = ((\!|(\tau'_a,\tau'_b)|\!) \otimes (\!|(\tau''_a,\tau''_b)|\!))$

$T_1 = \mathbb{M}\,0\,((\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!))$

$T_{1.1} = ((\!|\tau'_a|\!) \otimes (\!|\tau''_a|\!))$

$T_{1.11} = \llparenthesis \tau'_a \rrparenthesis$

$T_{1.12} = \llparenthesis \tau''_a \rrparenthesis$

$T_2 = \mathbb{M}\, 0\, (\llparenthesis \tau'_b \rrparenthesis \otimes \llparenthesis \tau''_b \rrparenthesis)$

$T_{2.1} = (\llparenthesis \tau'_b \rrparenthesis \otimes \llparenthesis \tau''_b \rrparenthesis)$

$T_{2.11} = \llparenthesis \tau'_b \rrparenthesis$

$T_{2.12} = \llparenthesis \tau''_b \rrparenthesis$

D6:

$$\dfrac{\dfrac{\dfrac{}{\begin{array}{c}.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash \\ \langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle : T_{0.61}\end{array}}}{\begin{array}{c}.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash \\ \mathsf{ret} \langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle : T_{0.6}\end{array}}}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash E_6 : T_{0.6}}$$

D5:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_2 : T_{2.1} \vdash P'_2 : T_{2.1}} \quad D6}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; , P'_2 : T_{2.1}, p'_{11} : T_{1.11}, p'_{12} : T_{1.12} \vdash \mathsf{let}\, !\langle\langle p'_{21}, p'_{22} \rangle\rangle = P'_2 \;\mathsf{in}\; E_6 : T_{0.6}}}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; , P'_2 : T_{2.1}, p'_{11} : T_{1.11}, p'_{12} : T_{1.12} \vdash E_5 : T_{0.6}}$$

D4:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; , P'_1 : T_{1.1} \vdash P'_1 : T_{1.1}} \quad D5}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_1 : T_{1.1}, P'_2 : T_{2.1} \vdash \mathsf{let}\, !\langle\langle p'_{11}, p'_{12} \rangle\rangle = P'_1 \;\mathsf{in}\; E_5 : T_{0.6}}}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_1 : T_{1.1}, P'_2 : T_{2.1} \vdash E_4 : T_{0.6}}$$

D3:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : \llparenthesis \tau_2 \rrparenthesis \vdash g'_2 p_2 : T_2} \quad D4}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : \llparenthesis \tau_2 \rrparenthesis, P'_1 : T_{1.1} \vdash \mathsf{bind}\, P'_2 = g'_2 p_2 \;\mathsf{in}\; E_4 : T_{0.6}}}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : \llparenthesis \tau_2 \rrparenthesis, P'_1 : T_{1.1} \vdash E_3 : T_{0.6}}$$

D2:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : \llparenthesis \tau_1 \rrparenthesis \vdash g'_1 p_1 : T_1} \quad D3}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : \llparenthesis \tau_1 \rrparenthesis, p_2 : \llparenthesis \tau_2 \rrparenthesis \vdash \mathsf{bind}\, P'_1 = g'_1 p_1 \;\mathsf{in}\; E_3 : T_{0.6}}}{.; .; .; f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : \llparenthesis \tau_1 \rrparenthesis, p_2 : \llparenthesis \tau_2 \rrparenthesis \vdash E_2 : T_{0.6}}$$

D1:

$$\dfrac{\dfrac{\dfrac{}{.; .; .; f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41} \vdash g_2 : T_{0.41}} \quad D2}{.; .; .; f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41}, p_1 : \llparenthesis \tau_1 \rrparenthesis, p_2 : \llparenthesis \tau_2 \rrparenthesis \vdash \mathsf{let}\, !\, g'_2 = g_2 \;\mathsf{in}\; E_2 : T_{0.6}}}{.; .; .; f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41}, p_1 : \llparenthesis \tau_1 \rrparenthesis, p_2 : \llparenthesis \tau_2 \rrparenthesis \vdash E_1 : T_{0.6}}$$

D0.1:

$$\dfrac{\dfrac{}{.;.;.;f:T_0;g_1:T_{0.31}\vdash g_1:T_{0.31}} \qquad D1}{\dfrac{.;.;.;f:T_0;g_1:T_{0.31},g_2:T_{0.41},p_1:(\!|\tau_1|\!),p_2:(\!|\tau_2|\!)\vdash \mathsf{let}\,!\,g_1'=g_1\;\mathsf{in}\;E_1:T_{0.6}}{.;.;.;f:T_0;g_1:T_{0.31},g_2:T_{0.41},p_1:(\!|\tau_1|\!),p_2:(\!|\tau_2|\!)\vdash E_0:T_{0.6}}}$$

D0:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{.;.;.;f:T_0;p:T_{0.51}\vdash p:T_{0.51}} \qquad D0.1}{.;.;.;f:T_0;g_1:T_{0.31},g_2:T_{0.41},p:T_{0.51}\vdash \mathsf{let}\,!\langle\!\langle p_1,p_2\rangle\!\rangle=p\;\mathsf{in}\;E_0:T_{0.6}}}{.;.;.;f:T_0;g_1:T_{0.31},g_2:T_{0.41}\vdash \lambda p.\,\mathsf{let}\,!\langle\!\langle p_1,p_2\rangle\!\rangle=p\;\mathsf{in}\;E_0:T_{0.5}}}{.;.;.;f:T_0;g_1:T_{0.31}\vdash .\lambda\_g_2.\lambda p.\,\mathsf{let}\,!\langle\!\langle p_1,p_2\rangle\!\rangle=p\;\mathsf{in}\;E_0:T_{0.4}}}{.;.;.;f:T_0;.\vdash \lambda\_g_1.\lambda\_g_2.\lambda p.\,\mathsf{let}\,!\langle\!\langle p_1,p_2\rangle\!\rangle=p\;\mathsf{in}\;E_0:T_0}}{.;.;.;.;.\vdash \mathsf{fix}f.\lambda\_g_1.\lambda\_g_2.\lambda p.\,\mathsf{let}\,!\langle\!\langle p_1,p_2\rangle\!\rangle=p\;\mathsf{in}\;E_0:T_0}$$

9. Sub:

$$\dfrac{\Sigma;\Gamma\vdash^{q}_{q'} e:\tau\rightsquigarrow e_a \qquad \tau<:\tau'}{\Sigma;\Gamma\vdash^{q}_{q'} e:\tau'\rightsquigarrow e_a}\;\text{Sub}$$

Main derivation:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!)\vdash e_a:[q]\,\mathbf{1}\multimap \mathbb{M}\,0\,([q'](\!|\tau|\!))} \qquad \dfrac{\tau<:\tau'}{.;.;.\vdash(\!|\tau|\!)<:(\!|\tau'|\!)}\;\text{Lemma 83}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!)\vdash e_a:[q]\,\mathbf{1}\multimap \mathbb{M}\,0\,([q'](\!|\tau'|\!))}\;\text{T-sub}$$

10. Super:

$$\dfrac{\Sigma;\Gamma,x:\tau_1\vdash^{q}_{q'} e:\tau\rightsquigarrow e_a \qquad \tau_1'<:\tau_1}{\Sigma;\Gamma,x:\tau_1'\vdash^{q}_{q'} e:\tau\rightsquigarrow e_a}\;\text{Super}$$

Main derivation:

$$\dfrac{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),x:(\!|\tau_1|\!)\vdash e_a:[q]\,\mathbf{1}\multimap \mathbb{M}\,0\,([q'](\!|\tau|\!)) \qquad \dfrac{\tau_1'<:\tau_1}{.;.;.\vdash(\!|\tau_1'|\!)<:(\!|\tau_1|\!)}\;\text{Lemma 83}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),x:(\!|\tau_1'|\!)\vdash e_a:[q]\,\mathbf{1}\multimap \mathbb{M}\,0\,([q'](\!|\tau|\!))}\;\text{T-weaken}$$

11. Relax:

$$\dfrac{\Sigma;\Gamma\vdash^{p}_{p'} e:\tau\rightsquigarrow e_a \qquad q\geqslant p \qquad q-p\geqslant q'-p'}{\Sigma;\Gamma\vdash^{q}_{q'} e:\tau\rightsquigarrow \lambda o.E_0}\;\text{Relax}$$

where

$E_0 = \mathsf{release}\,-=o\;\mathsf{in}\;E_1$

$E_1 = \mathsf{bind}\,a=\mathsf{store}()\;\mathsf{in}\;E_2$

$E_2 = \mathsf{bind}\,b=e_a\;a\;\mathsf{in}\;E_3$

$E_3 = \mathsf{release}\, c = b \;\mathsf{in}\; \mathsf{store}\, c$

D2:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);b:[p'](\!|\tau|\!) \vdash b:[p'](\!|\tau|\!)}}{\dfrac{.;.;.;(\!|\Sigma|\!);c:(\!|\tau|\!) \vdash \mathsf{store}\, c : \mathbb{M}(q-p+p')\,([q-p+p'](\!|\tau|\!))}{.;.;.;(\!|\Sigma|\!);b:[p'](\!|\tau|\!) \vdash E_3 : \mathbb{M}(q-p)\,([q-p+p'](\!|\tau|\!))}}$$

D1.2:

$$\dfrac{}{.;.;.;(\!|\Sigma|\!);a:[p]\,\mathbf{1} \vdash a:[p]\,\mathbf{1}}$$

D1.1:

$$\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!) \vdash e_a : [p]\,\mathbf{1} \multimap \mathbb{M}\,0\,([p'](\!|\tau|\!))}\;\text{IH}$$

D1:

$$\dfrac{\dfrac{D1.1 \qquad D1.2}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),a:[p]\,\mathbf{1} \vdash e_a\; a : \mathbb{M}\,0\,([p'](\!|\tau|\!))} \qquad D2}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),a:[p]\,\mathbf{1} \vdash E_2 : \mathbb{M}(q-p)\,([q-p+p'](\!|\tau|\!))}$$

D0:

$$\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);. \vdash \mathsf{store}() : \mathbb{M}\,p\,([p]\,\mathbf{1})} \qquad D1}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!) \vdash E_1 : \mathbb{M}(q)\,([q-p+p'](\!|\tau|\!))}$$

D0.0:

$$\dfrac{\dfrac{\dfrac{}{q' \leqslant q-p+p'}\;\text{Given}}{.;.;. \vdash ([q-p+p'](\!|\tau|\!)) <: ([q'](\!|\tau|\!))}}{.;.;. \vdash \mathbb{M}\,0\,([q-p+p'](\!|\tau|\!)) <: \mathbb{M}\,0\,([q'](\!|\tau|\!))}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);o:[q]\,\mathbf{1} \vdash o:[q]\,\mathbf{1}} \qquad D0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),o:[q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,([q-p+p'](\!|\tau|\!))} \qquad D0.0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!),o:[q]\,\mathbf{1} \vdash E_0 : \mathbb{M}\,0\,([q'](\!|\tau|\!))}\;\text{T-sub}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma|\!) \vdash \lambda o.E_0 : [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q'](\!|\tau|\!))}$$

12. Let:

$$\dfrac{\Sigma;\Gamma_1 \vdash_{p}^{q-K_1^{let}} e_1 : \tau_1 \rightsquigarrow e_{a1} \qquad \Sigma;\Gamma_2, x:\tau_1 \vdash_{q'+K_3^{let}}^{p-K_2^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma;\Gamma_1,\Gamma_2 \vdash_{q'}^{q} \mathsf{let}\; x = e_1 \; in \; e_2 : \tau \rightsquigarrow E_t}\;\text{Let}$$

where

$E_t = \lambda u.E_0$

$E_0 = \mathsf{release} - = u \;\mathsf{in}\; E_1$

$E_1 = \mathsf{bind} - = \uparrow^{K_1^{let}} \;\mathsf{in}\; E_2$

$E_2 = \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; E_3$

$E_3 = \textsf{bind}\, b = e_{a1}\ a \ \textsf{in}\ E_4$

$E_4 = \textsf{release}\, x = b \ \textsf{in}\ E_5$

$E_5 = \textsf{bind}\, - = \uparrow^{K_2^{let}} \ \textsf{in}\ E_6$

$E_6 = \textsf{bind}\, c = \textsf{store}() \ \textsf{in}\ E_7$

$E_7 = \textsf{bind}\, d = e_{a2}\ c \ \textsf{in}\ E_8$

$E_8 = \textsf{release}\, f = d \ \textsf{in}\ E_9$

$E_9 = \textsf{bind}\, - = \uparrow^{K_3^{let}} \ \textsf{in}\ E_{10}$

$E_{10} = \textsf{bind}\, g = \textsf{store}\, f \ \textsf{in}\ \textsf{ret}\, g$

$T_0 = [q]\,\mathbf{1} \multimap \mathbb{M}\,0\,([q'](\!|\tau|\!))$

$T_{0.1} = [q]\,\mathbf{1}$

$T_{0.2} = \mathbb{M}\,0\,([q'](\!|\tau|\!))$

$T_{0.3} = \mathbb{M}\,q\,([q'](\!|\tau|\!))$

$T_{0.4} = \mathbb{M}(q - K_1^{let})\,([q'](\!|\tau|\!))$

$T_{0.5} = \mathbb{M}(q - K_1^{let})\,([q - K_1^{let}]\,\mathbf{1})$

$T_{0.51} = [q - K_1^{let}]\,\mathbf{1}$

$T_{0.6} = \mathbb{M}\,0\,[p]\,(\!|\tau_1|\!)$

$T_{0.61} = [p]\,(\!|\tau_1|\!)$

$T_{0.7} = \mathbb{M}\,p\,([q'](\!|\tau|\!))$

$T_{0.8} = \mathbb{M}(p - K_2^{let})\,([q'](\!|\tau|\!))$

$T_{0.9} = \mathbb{M}(p - K_2^{let})\,([(p - K_2^{let})]\,\mathbf{1})$

$T_{0.91} = [(p - K_2^{let})]\,\mathbf{1}$

$T_1 = \mathbb{M}\,0\,[(q' + {K_3}^{let})]\,(\!|\tau|\!)$

$T_{1.1} = [(q' + {K_3}^{let})]\,(\!|\tau|\!)$

$T_{1.2} = \mathbb{M}(q' + {K_3}^{let})\,([q'](\!|\tau|\!))$

$T_{1.3} = \mathbb{M}\,q'\,([q'](\!|\tau|\!))$

D10:

$$\frac{}{.;.;.;(\!|\Sigma|\!); g : [q']\,(\!|\tau|\!) \vdash \textsf{ret}\, g : \mathbb{M}\,0\,[q']\,(\!|\tau|\!)}$$

D9:

$$\frac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!); f : (\!|\tau|\!) \vdash \textsf{store}\, f : T_{1.3}} \qquad D10}{.;.;.;(\!|\Sigma|\!); f : (\!|\tau|\!) \vdash \textsf{bind}\, g = \textsf{store}\, f \ \textsf{in}\ \textsf{ret}\, g : T_{1.3}}}{.;.;.;(\!|\Sigma|\!); f : (\!|\tau|\!) \vdash E_{10} : T_{1.3}}$$

D8:

$$\frac{\dfrac{\dfrac{}{.;.;.;. \vdash \uparrow^{K_3^{let}} : \mathbb{M}\,K_3^{let}\,\mathbf{1}} \qquad D9}{.;.;.;(\!|\Sigma|\!); f : (\!|\tau|\!) \vdash \textsf{bind}\, - = \uparrow^{K_3^{let}} \ \textsf{in}\ E_{10} : T_{1.2}}}{.;.;.;(\!|\Sigma|\!); f : (\!|\tau|\!) \vdash E_9 : T_{1.2}}$$

D7:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);d:T_{1.1}\vdash d:T_{1.1}} \qquad D8}{.;.;.;(\!|\Sigma|\!);d:T_{1.1}\vdash \mathsf{release}\, f = d \;\mathsf{in}\; E_9 : T_{0.2}}}{.;.;.;(\!|\Sigma|\!);d:T_{1.1}\vdash E_8 : T_{0.2}}$$

D6:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!),c:T_{0.91}\vdash e_{a2}\ c:T_1} \qquad D7}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!),c:T_{0.91}\vdash \mathsf{bind}\, d = e_{a2}\ c \;\mathsf{in}\; E_8 : T_{0.2}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!),c:T_{0.91}\vdash E_7 : T_{0.2}}$$

D5:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);.\vdash \mathsf{store}() : T_{0.9}} \qquad D6}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!)\vdash \mathsf{bind}\, c = \mathsf{store}() \;\mathsf{in}\; E_7 : T_{0.8}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!)\vdash E_6 : T_{0.8}}$$

D4:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);.\vdash \uparrow^{K_2^{let}} : \mathbb{M}\, K_2^{let}\, \mathbf{1}} \qquad D5}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!)\vdash \mathsf{bind} - = \uparrow^{K_2^{let}} \;\mathsf{in}\; E_6 : T_{0.7}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!)\vdash E_5 : T_{0.7}}$$

D3:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);b:T_{0.61}\vdash b:T_{0.61}} \qquad D4}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!),b:T_{0.61}\vdash \mathsf{release}\, x = b \;\mathsf{in}\; E_5 : T_{0.2}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_2|\!),b:T_{0.61}\vdash E_4 : T_{0.2}}$$

D2:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),a:T_{0.51}\vdash e_{a1}\ a:T_{0.6}} \qquad D3}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!),a:T_{0.51}\vdash \mathsf{bind}\, b = e_{a1}\ a \;\mathsf{in}\; E_4 : T_{0.2}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!),a:T_{0.51}\vdash E_3 : T_{0.2}}$$

D1:

$$\dfrac{\dfrac{\overline{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash \mathsf{store}() : T_{0.5}} \qquad D2}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; E_3 : T_{0.4}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash E_2 : T_{0.4}}$$

D0:

$$\dfrac{\dfrac{\overline{.;.;.;.\vdash \uparrow^{K_1^{let}} : \mathbb{M}\, K_1^{let}\, \mathbf{1}} \qquad D1}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash \mathsf{bind} - = \uparrow^{K_1^{let}} \;\mathsf{in}\; E_2 : T_{0.3}}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash E_1 : T_{0.3}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!),u:T_{0.1}\vdash u:T_{0.1}}\qquad D0}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!),u:T_{0.1}\vdash \mathsf{release}\,-=u\;\mathsf{in}\;E_1:T_{0.2}}}{\dfrac{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!),u:T_{0.1}\vdash E_0:T_{0.2}}{.;.;.;(\!|\Sigma|\!);(\!|\Gamma_1|\!),(\!|\Gamma_2|\!)\vdash \lambda u.E_0:T_0}}$$

13. Pair:

$$\dfrac{}{\Sigma;x_1:\tau_1,x_2:\tau_2\vdash_q^{q+K^{pair}}(x_1,x_2):(\tau_1,\tau_2)\rightsquigarrow E_t}\;\text{pair}$$

where

$E_t=\lambda u.E_0$

$E_0=\mathsf{release}\,-=u\;\mathsf{in}\;E_1$

$E_1=\mathsf{bind}\,-=\uparrow^{K^{pair}}\;\mathsf{in}\;E_2$

$E_2=\mathsf{bind}\,a=\mathsf{store}(x_1,x_2)\;\mathsf{in}\;\mathsf{ret}\,a$

$T_0=[(q+K^{pair})]\,\mathbf{1}\multimap\mathbb{M}\,0\,([q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!))$

$T_{0.1}=[(q+K^{pair})]\,\mathbf{1}$

$T_{0.2}=\mathbb{M}\,0\,([q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!))$

$T_{0.3}=\mathbb{M}\,(q+K^{pair})\,([q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!))$

$T_{0.4}=\mathbb{M}\,q\,([q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!))$

D2:

$$\dfrac{}{.;.;.;(\!|\Sigma|\!);a:[q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!)\vdash \mathsf{ret}\,a:\mathbb{M}\,0\,[q]\,(\!|\tau_1|\!)\otimes(\!|\tau_2|\!)}$$

D1:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash \mathsf{store}(x_1,x_2):T_{0.4}}\qquad D2}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash \mathsf{bind}\,a=\mathsf{store}(x_1,x_2)\;\mathsf{in}\;\mathsf{ret}\,a:T_{0.4}}}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash E_2:T_{0.4}}$$

D0:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);.\vdash\uparrow^{K^{pair}}:\mathbb{M}\,K^{pair}\,\mathbf{1}}\qquad D1}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash \mathsf{bind}\,-=\uparrow^{K^{pair}}\;\mathsf{in}\;E_2:T_{0.3}}}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash E_1:T_{0.3}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!),u:T_{0.1}\vdash u:T_{0.1}}\qquad D0}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!),u:T_{0.1}\vdash \mathsf{release}\,-=u\;\mathsf{in}\;E_1:T_{0.2}}}{\dfrac{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!),u:T_{0.1}\vdash E_0:T_{0.2}}{.;.;.;(\!|\Sigma|\!);x_1:(\!|\tau_1|\!),x_2:(\!|\tau_2|\!)\vdash \lambda u.E_0:T_0}}$$

14. MatP:

$$\frac{\tau = (\tau_1, \tau_2) \qquad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash^{q-K_1^{matP}}_{q'+K_2^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash^{q}_{q'} match\ x\ with\ (x_1, x_2) \to e : \tau' \rightsquigarrow E_t} \text{matP}$$

where

$E_t = \lambda u.E_0$

$E_0 = \mathsf{release} - = u \mathsf{\ in\ } E_1$

$E_1 = \mathsf{bind} - = \uparrow^{K_1^{matP}} \mathsf{\ in\ } E_2$

$E_2 = \mathsf{let}\langle\langle x_1, x_2\rangle\rangle = x \mathsf{\ in\ } E_3$

$E_3 = \mathsf{bind}\, a = \mathsf{store}() \mathsf{\ in\ } E_4$

$E_4 = \mathsf{bind}\, b = e_t\ a \mathsf{\ in\ } E_5$

$E_5 = \mathsf{release}\, c = b \mathsf{\ in\ } E_6$

$E_6 = \mathsf{bind} - = \uparrow^{K_2^{matP}} \mathsf{\ in\ } E_7$

$E_7 = \mathsf{bind}\, d = \mathsf{store}\, c \mathsf{\ in\ ret}\, d$

$T_0 = [q]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q']\, (\!|\tau'|\!))$

$T_{0.1} = [q]\, \mathbf{1}$

$T_{0.2} = \mathbb{M}\, 0\, ([q']\, (\!|\tau'|\!))$

$T_{0.3} = \mathbb{M}\, q\, ([q']\, (\!|\tau'|\!))$

$T_{0.4} = \mathbb{M}(q - K_1^{matP})\, ([q']\, (\!|\tau'|\!))$

$T_{0.5} = \mathbb{M}(q - K_1^{matP})\, ([(q - K_1^{matP})]\, \mathbf{1})$

$T_{0.51} = [(q - K_1^{matP})]\, \mathbf{1}$

$T_{0.6} = \mathbb{M}\, 0\, ([q' + k_2^{matP}]\, (\!|\tau'|\!))$

$T_{0.61} = [(q' + k_2^{matP})]\, (\!|\tau'|\!)$

$T_{0.7} = \mathbb{M}(q' + K_2^{matP})\, [q']\, (\!|\tau'|\!)$

$T_{0.71} = [q']\, (\!|\tau'|\!)$

$T_{0.8} = \mathbb{M}\, q'\, ([q']\, (\!|\tau'|\!))$

D7:

$$\frac{}{.; .; .; (\!|\Sigma|\!); d : [q']\, (\!|\tau'|\!) \vdash \mathsf{ret}\, d : \mathbb{M}\, 0\, [q']\, (\!|\tau'|\!)}$$

D6:

$$\frac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash \mathsf{store}\, c : \mathbb{M}\, q'\, [q']\, (\!|\tau'|\!)} \qquad D7}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash \mathsf{bind}\, d = \mathsf{store}\, c \mathsf{\ in\ ret}\, d : T_{0.8}}}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash E_7 : T_{0.8}}$$

D5:

$$\frac{\dfrac{\dfrac{}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash \uparrow^{K_2^{matP}} : \mathbb{M}\, K_2^{matP}\, \mathbf{1}} \qquad D6}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash \mathsf{bind} - = \uparrow^{K_2^{matP}} \mathsf{\ in\ } E_7 : T_{0.7}}}{.; .; .; (\!|\Sigma|\!); c : (\!|\tau'|\!) \vdash E_6 : T_{0.7}}$$

D4:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; b : T_{0.61} \vdash b : T_{0.61}} \quad D5}{.;.;.;\llparenthesis\Sigma\rrparenthesis; b : T_{0.61} \vdash \mathsf{release}\, c = b \,\mathsf{in}\, E_6 : T_{0.2}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; b : T_{0.61} \vdash E_5 : T_{0.2}}$$

D3:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x_1 : \llparenthesis\tau_1\rrparenthesis, x_2 : \llparenthesis\tau_2\rrparenthesis, a : T_{0.51} \vdash e_t\ a : T_{0.6}} \quad D4}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x_1 : \llparenthesis\tau_1\rrparenthesis, x_2 : \llparenthesis\tau_2\rrparenthesis, a : T_{0.51} \vdash \mathsf{bind}\, b = e_t\ a \,\mathsf{in}\, E_5 : T_{0.2}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x_1 : \llparenthesis\tau_1\rrparenthesis, x_2 : \llparenthesis\tau_2\rrparenthesis, a : T_{0.51} \vdash E_4 : T_{0.2}}$$

D2:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; . \vdash \mathsf{store}() : T_{0.5}} \quad D3}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x_1 : \llparenthesis\tau_1\rrparenthesis, x_2 : \llparenthesis\tau_2\rrparenthesis \vdash \mathsf{bind}\, a = \mathsf{store}() \,\mathsf{in}\, E_4 : T_{0.4}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x_1 : \llparenthesis\tau_1\rrparenthesis, x_2 : \llparenthesis\tau_2\rrparenthesis \vdash E_3 : T_{0.4}}$$

D1:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; x : \llparenthesis\tau\rrparenthesis \vdash x : \llparenthesis\tau\rrparenthesis} \quad D2}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis \vdash \mathsf{let}\langle\!\langle x_1, x_2\rangle\!\rangle = x \,\mathsf{in}\, E_3 : T_{0.4}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis \vdash E_2 : T_{0.4}}$$

D0:

$$\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; . \vdash \uparrow^{K_1^{matP}} : \mathbb{M}\, K_1^{matP}\, \mathbf{1}} \quad D1}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis \vdash \mathsf{bind} - = \uparrow^{K_1^{matP}} \,\mathsf{in}\, E_2 : T_{0.3}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis \vdash E_1 : T_{0.3}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis, u : T_{0.1} \vdash u : T_{0.1}} \quad D0}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis, u : T_{0.1} \vdash \mathsf{release} - = u \,\mathsf{in}\, E_1 : T_{0.2}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis, u : T_{0.1} \vdash E_0 : T_{0.2}}}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau\rrparenthesis \vdash \lambda u.E_0 : T_0}$$

15. Augment:

$$\dfrac{\Sigma; \Gamma \vdash^{q}_{q'} e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash^{q}_{q'} e : \tau \rightsquigarrow e_a} \text{ Augment}$$

Main derivation:

$$\dfrac{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis \vdash e_a : [q]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q']\llparenthesis\tau\rrparenthesis)}{.;.;.;\llparenthesis\Sigma\rrparenthesis; \llparenthesis\Gamma\rrparenthesis, x : \llparenthesis\tau'\rrparenthesis \vdash e_a : [q]\, \mathbf{1} \multimap \mathbb{M}\, 0\, ([q']\llparenthesis\tau\rrparenthesis)} \text{ T-weaken}$$

$\square$

Lemma 83 (Subtyping preservation)**.** $\forall \tau, \tau'$.
$\tau <: \tau' \implies \llparenthesis\tau\rrparenthesis <: \llparenthesis\tau'\rrparenthesis$

Proof. Proof by induction on the $\tau <: \tau'$ relation

1. Base:

$$\frac{}{\mathsf{b} <: \mathsf{b}}$$

Main derivation:

$$\frac{}{.;. \vdash !\mathsf{b} <: !\mathsf{b}}$$

2. Pair:

$$\frac{\tau_1 <: \tau_1' \qquad \tau_2 <: \tau_2'}{(\tau_1, \tau_2) <: (\tau_1', \tau_2')}$$

Main derivation:

$$\frac{\dfrac{}{(\!|\tau_1|\!) <: (\!|\tau_1'|\!)}\text{IH1} \qquad \dfrac{}{(\!|\tau_2|\!) <: (\!|\tau_2'|\!)}\text{IH2}}{((\!|\tau_1|\!) \otimes (\!|\tau_2|\!)) <: ((\!|\tau_1'|\!) \otimes (\!|\tau_2'|\!))}$$

3. List:

$$\frac{\tau_1 <: \tau_2 \qquad \vec{p} \geqslant \vec{q}}{L^{\vec{p}}\tau_1 <: L^{\vec{q}}\tau_2}$$

Main derivation:

$$\frac{\dfrac{\dfrac{\dfrac{}{\vec{q} \leqslant \vec{p}}\text{Given}}{\phi(\vec{q}, s) \leqslant \phi(\vec{p}, s)}}{.; s \vdash [\phi(\vec{p}, s)]\,\mathbf{1} <: [\phi(\vec{q}, s)]\,\mathbf{1}} \qquad \dfrac{\dfrac{}{.; s \vdash (\!|\tau_1|\!) <: (\!|\tau_2|\!)}\text{IH}}{.; s \vdash L^s(\!|\tau_1|\!) <: L^s(\!|\tau_2|\!)}}{\dfrac{.; s \vdash ([\phi(\vec{p}, s)]\,\mathbf{1} \otimes L^s(\!|\tau_1|\!)) <: ([\phi(\vec{q}, s)]\,\mathbf{1} \otimes L^s(\!|\tau_2|\!))}{.; . \vdash \exists s.([\phi(\vec{p}, s)]\,\mathbf{1} \otimes L^s(\!|\tau_1|\!)) <: \exists s.([\phi(\vec{q}, s)]\,\mathbf{1} \otimes L^s(\!|\tau_2|\!))}}$$

□

2.5.2 Cross-language model: RAMLU to λ-Amor

Definition 84 (Logical relation for RAMLU to λ-Amor).

$$
\begin{array}{lll}
\lfloor unit \rfloor_{\mathcal{V}}^{H} & \triangleq & \{(T, {}^{s}v, {}^{t}v) \mid {}^{s}v \in \llbracket unit \rrbracket \wedge {}^{t}v \in \llbracket \mathbf{1} \rrbracket \wedge {}^{s}v = {}^{t}v\} \\
\lfloor \mathsf{b} \rfloor_{\mathcal{V}}^{H} & \triangleq & \{(T, {}^{s}v, !{}^{t}v) \mid {}^{s}v \in \llbracket \mathsf{b} \rrbracket \wedge {}^{t}v \in \llbracket \mathsf{b} \rrbracket \wedge {}^{s}v = {}^{t}v\} \\
\lfloor (\tau_1, \tau_2) \rfloor_{\mathcal{V}}^{H} & \triangleq & \{(T, \ell, \langle\langle {}^{t}v_1, {}^{t}v_2 \rangle\rangle) \mid H(\ell) = ({}^{s}v_1, {}^{s}v_2) \wedge (T, {}^{s}v_1, {}^{t}v_1) \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \wedge (T, {}^{s}v_2, {}^{t}v_2) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}\} \\
\lfloor L^{\vec{q}} \tau \rfloor_{\mathcal{V}}^{H} & \triangleq & \{(T, \ell_s, \langle\langle (), l_t \rangle\rangle) \mid (T, \ell_s, l_t) \in \lfloor L\ \tau \rfloor_{\mathcal{V}}^{H}\} \\
where & & \\
\lfloor L\ \tau \rfloor_{\mathcal{V}}^{H} & \triangleq & \{(T, NULL, nil)\}\} \cup \\
& & \{(T, \ell, {}^{t}v :: l_t) \mid H(\ell) = ({}^{s}v, \ell_s) \wedge (T, {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}} \wedge (T, \ell_s, l_t) \in \lfloor L\ \tau \rfloor_{\mathcal{V}}\} \\
\lfloor \tau_1 \xrightarrow{q/q'} \tau_2 \rfloor_{\mathcal{V}'}^{H} & \triangleq & \{(T, f(x) = e_s, \mathsf{fix} f.\lambda u.\lambda x.e_t) \mid \forall {}^{s}v', {}^{t}v', T' < T\ . \\
& & (T', {}^{s}v', {}^{t}v') \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \implies (T', e_s, e_t[()/u][{}^{t}v'/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto {}^{s}v'\}, H}\} \\
\\
\lfloor \tau \rfloor_{\mathcal{E}}^{V,H} & \triangleq & \{(T, e_s, e_t) \mid \forall H', {}^{s}v, p, p', t < T\ .\ V, H \vdash_{p'}^{p} e_s \Downarrow_t {}^{s}v, H' \implies \\
& & \exists {}^{t}v_t, {}^{t}v_f, J. e_t \Downarrow_{-} {}^{t}v_t \Downarrow_{-}^{J} {}^{t}v_f \wedge (T - t, {}^{s}v, {}^{t}v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J\}
\end{array}
$$

Definition 85 (Interpretation of typing context).

$$\lfloor \Gamma \rfloor_{\mathcal{V}}^{H} \quad = \quad \{(T, V, \delta_t) \mid \forall x : \tau \in dom(\Gamma).(T, V(x), \delta_t(x)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}\}$$

Definition 86 (Interpretation of function context).

$$\lfloor \Sigma \rfloor_{\mathcal{V}}^{H} \quad = \quad \{(T, \delta_{sf}, \delta_{tf}) \mid (\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in dom(\Sigma).(T, \delta_{sf}(f)\ \delta_{sf}, \delta_{tf}(f)\ \delta_{tf}) \in \lfloor (\tau_1 \xrightarrow{q/q'} \tau_2) \rfloor_{\mathcal{V}'}^{H})\}$$

Lemma 87 (Monotonicity for values). $\forall {}^{s}v, {}^{t}v, T, \tau, H.$
$(T, {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H} \implies \forall\ T' \leqslant T\ .(T', {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$

Proof. Given: $(T, {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$
To prove: $\forall\ T' \leqslant T\ .(T', {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$

This means given some $T' \leqslant T$ it suffices to prove that
$(T', {}^{s}v, {}^{t}v) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$
By induction on τ

1. $\tau = unit$:

   In this case we are given that $(T, {}^{s}v, {}^{t}v) \in \lfloor unit \rfloor_{\mathcal{V}}^{H}$

   and we need to prove $(T', {}^{s}v, {}^{t}v) \in \lfloor unit \rfloor_{\mathcal{V}}^{H}$

   We get the desired trivially from Definition 84

2. $\tau = \mathsf{b}$:

   In this case we are given that $(T, {}^{s}v, !{}^{t}v') \in \lfloor \mathsf{b} \rfloor_{\mathcal{V}}^{H}$

   and we need to prove $(T', {}^{s}v, !{}^{t}v') \in \lfloor \mathsf{b} \rfloor_{\mathcal{V}}^{H}$

   We get the desired trivially from Definition 84

3. $\tau = L^{\vec{p}} \tau'$:

   In this case we are given that $(T, {}^{s}v, {}^{t}v) \in \lfloor L^{\vec{p}} \tau' \rfloor_{\mathcal{V}}^{H}$

   Here let ${}^{s}v = \ell_s$ and ${}^{t}v = \langle\langle (), {}^{t}v_h :: l_t \rangle\rangle$

and we have $(T, \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$ (MV-L1)

And we need to prove $(T', \ell_s, {}^t v_h :: l_t) \in \lfloor L^{\vec{p}}\tau' \rfloor_{\mathcal{V}}^H$

Therefore it suffices to prove that $(T', \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$

We induct on $(T, \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$

- $(T, NULL, nil) \in \lfloor L^{\vec{p}}\tau' \rfloor_{\mathcal{V}}^H$:
  In this case we need to prove that $(T', NULL, nil) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$
  We get this directly from Definition 84
- $(T, \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$:
  Since from (MV-L1) we are given that $(T, \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$
  therefore from Definition 84 we have
  $H(\ell_s) = ({}^s v_h, \ell_{st}) \wedge (T, {}^s v_h, {}^t v_h) \in \lfloor \tau' \rfloor_{\mathcal{V}} \wedge (T, \ell_{st}, l_t) \in \lfloor L\ \tau' \rfloor_{\mathcal{V}}$ (MV-L2)

  In this case we need to prove that $(T', \ell_s, {}^t v_h :: l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^H$
  From Definition 84 it further it suffices to prove that
  - $H(\ell_s) = ({}^s v_h, \ell_{st})$:
    Directly from (MV-L2)
  - $(T', {}^s v_h, {}^t v_h) \in \lfloor \tau' \rfloor_{\mathcal{V}}$:
    From (MV-L2) and outer induction
  - $(T', \ell_{st}, l_t) \in \lfloor L\ \tau' \rfloor_{\mathcal{V}}$:
    From (MV-L2) and inner induction

4. $\tau = (\tau_1, \tau_2)$:
   In this case we are given that $(T, \ell, ({}^t v_1, {}^t v_2)) \in \lfloor (\tau_1, \tau_2) \rfloor_{\mathcal{V}}^H$

   This means from Definition 84 we have
   $H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}$ (MV-P0)

   and we need to prove $(T', \ell, ({}^t v_1, {}^t v_2)) \in \lfloor (\tau_1, \tau_2) \rfloor_{\mathcal{V}}^H$
   Similarly from Definition 84 it suffices to prove that
   $H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T', {}^s v_1, {}^t v_1) \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \wedge (T', {}^s v_2, {}^t v_2) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}$
   We get this directly from (MV-P0), IH1 and IH2

□

Lemma 88 (Monotonicity for functions)**.** $\forall {}^s v, {}^t v, T, \tau, H.$
$(T, f(x) = e_s, \mathsf{fix} f.\lambda u.\lambda x.e_t) \in \lfloor \tau_1 \xrightarrow{q/q'} \tau_2 \rfloor_{\mathcal{V}'}^H \implies \forall\ T' \leqslant T\ .(T', f(x) = e_s, \mathsf{fix} f.\lambda u.\lambda x.e_t) \in \lfloor \tau_1 \xrightarrow{q/q'} \tau_2 \rfloor_{\mathcal{V}'}^H$

Proof. We need to prove that $(T', f(x) = e_s, \mathsf{fix} f.\lambda u.\lambda x.e_t) \in \lfloor \tau_1 \xrightarrow{q/q'} \tau_2 \rfloor_{\mathcal{V}'}^H$

This means from Definition 84 it suffices to prove that
$\forall {}^s v', {}^t v', T'' < T'\ .(T'', {}^s v', {}^t v') \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \implies (T'', e_s, e_t[()/u][{}^t v'/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$

This means given some ${}^s v', {}^t v', T'' < T'$ s.t $(T'', {}^s v', {}^t v') \in \lfloor \tau_1 \rfloor_{\mathcal{V}}$ it suffices to prove that

$(T'', e_s, e_t[()/u][{}^t v'/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$ (MF0)

Since we are given that $(T, f(x) = e_s, \mathsf{fix} f.\lambda u.\lambda x.e_t) \in \lfloor \tau_1 \xrightarrow{q/q'} \tau_2 \rfloor_{\mathcal{V}'}^{H}$ therefore from Definition 84 we have

$\forall {}^s v'_1, {}^t v'_1, T'_1 {<} T .(T'_1, {}^s v'_1, {}^t v'_1) \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \implies (T'_1, e_s, e_t[()/u][{}^t v'_1/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto {}^s v'_1\}, H}$

Instantiating with the given ${}^s v', {}^t v', T''$ we get the desired

□

Lemma 89 (Monotonicity for expressions)**.** $\forall e_s, e_t, T, \tau, H.$
$(T, e_s, e_t) \in \lfloor \tau \rfloor_{\mathcal{E}}^{H} \implies \forall\, T' {\leqslant} T .(T', e_s, e_t) \in \lfloor \tau \rfloor_{\mathcal{E}}^{H}$

Proof. To prove: $(T', e_s, e_t) \in \lfloor \tau \rfloor_{\mathcal{E}}^{H}$

This means from Definition 84 it suffices to prove that

$\forall H', {}^s v, p, p', t \;{<}T' \;.\; V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_t \Downarrow_{-} {}^t v_t \Downarrow^{J}_{-} {}^t v_f \wedge (T' - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

This means given some $H', {}^s v, p, p', t \;{<}T'$ s.t $V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$\exists {}^t v_t, {}^t v_f, J. e_t \Downarrow_{-} {}^t v_t \Downarrow^{J}_{-} {}^t v_f \wedge (T' - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (ME0)

Since we are given that $(T, e_s, e_t) \in \lfloor \tau \rfloor_{\mathcal{E}}^{H}$ therefore again from Definition 84 we know that

$\forall H', {}^s v, p, p', t \;{<}T \;.\; V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_t \Downarrow_{-} {}^t v_t \Downarrow^{J}_{-} {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

Instantiating with the given $H', {}^s v, p, p', t$ and using Lemma 87 we get the desired

□

Lemma 90 (Monotonicity for Γ)**.** $\forall {}^s v, {}^t v, T, \tau, H.$
$(T, V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{H} \implies \forall\, T' {\leqslant} T .(T', V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{H}$

Proof. To prove: $(T', V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{H}$

From Definition 85 it suffices to prove that

$\forall x : \tau \in dom(\Gamma).(T', V(x), \delta_t(x)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$

This means given some $x : \tau \in dom(\Gamma)$ it suffices to prove that

$(T', V(x), \delta_t(x)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$

Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{H}$
therefore from Definition 85 we have
$\forall x : \tau \in dom(\Gamma).(T, V(x), \delta_t(x)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$
Instantiating it with the given x and using Lemma 87 we get the desired □

Lemma 91 (Monotonicity for Σ)**.** $\forall {}^s v, {}^t v, T, \tau, H.$
$(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}}^{H} \implies \forall\, T' {\leqslant} T .(T', \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}}^{H}$

Proof. To prove: $(T', \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}}^{H}$

From Definition 86 it suffices to prove that

$(\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in dom(\Sigma).(T', \delta_{sf}(f)\ \delta_{sf}, \delta_{tf}(f)\ \delta_{tf}) \in \lfloor (\tau_1 \xrightarrow{q/q'} \tau_2) \rfloor_{\mathcal{V}'}^{H})$

This means given some $f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in dom(\Sigma)$ it suffices to prove that

$(T', \delta_{sf}(f)\ \delta_{sf}, \delta_{tf}(f)\ \delta_{tf}) \in \lfloor (\tau_1 \xrightarrow{q/q'} \tau_2) \rfloor_{\mathcal{V}'}^{H}$

Since we are given that $(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}}^{H}$
therefore from Definition 85 we have
$(\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in dom(\Sigma).(T, \delta_{sf}(f)\ \delta_{sf}, \delta_{tf}(f)\ \delta_{tf}) \in \lfloor (\tau_1 \xrightarrow{q/q'} \tau_2) \rfloor_{\mathcal{V}'}^{H})$
Instantiating it with the given f and using Lemma 88 we get the desired □

Theorem 92 (Fundamental theorem)**.** $\forall \Sigma, \Gamma, q, q', \tau, e_s, e_t, I, V, H, \delta_t, \delta_{sf}, \delta_{tf}, T.$
$\Sigma; \Gamma \vdash^q_{q'} e_s : \tau \rightsquigarrow e_t \ \wedge$
$(T, V, \delta_t) \in \lfloor \Gamma \rceil^H_{\mathcal{V}} \ \wedge \ (T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil^H_{\mathcal{V}}$
$\Longrightarrow$
$(T, e_s\delta_{sf}, e_t \ () \ \delta_t\delta_{tf}) \in \lfloor \tau \rceil^{V,H}_{\mathcal{E}}$

Proof. Proof by induction on $\Sigma; \Gamma \vdash^q_{q'} e_s : \tau \rightsquigarrow e_t$

1. unit:

$$\frac{}{\Sigma; . \vdash^{q+K^{unit}}_{q} () : unit \rightsquigarrow E_t} \text{unit}$$

where

$E_t = \lambda u.\mathsf{release} - = u \ \mathsf{in} \ \mathsf{bind} - = \uparrow^{K^{unit}} \ \mathsf{in} \ \mathsf{bind} \ a = \mathsf{store}() \ \mathsf{in} \ \mathsf{ret}(a)$

$E'_t = \mathsf{release} - = u \ \mathsf{in} \ \mathsf{bind} - = \uparrow^{K^{unit}} \ \mathsf{in} \ \mathsf{bind} \ a = \mathsf{store}() \ \mathsf{in} \ \mathsf{ret}(a)$

To prove: $(T, x\delta_{sf}, E_t \ () \ \delta_t\delta_{tf}) \in \lfloor \tau \rceil^{V,H}_{\mathcal{E}}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t$ s.t $V, H \vdash^r_{r'} () \Downarrow_t (), H$. From (E:Unit) we know that $t = 1$

Therefore it suffices to prove that

(a) $\exists {}^t v_t, {}^t v_f, J.E_t \ () \Downarrow_- {}^t v_t \Downarrow^J_- {}^t v_f \ \wedge \ (T - 1, (), {}^t v_f) \in \lfloor unit \rceil_{\mathcal{V}}$:
We choose ${}^t v_t, {}^t v_f, J$ as $E'_t, (), K^{unit}$ respectively
Since from E-app we know that $E_t \Downarrow E'_t$, also since $E'_t \Downarrow^{K^{unit}} ()$ (from E-release, E-bind, E-store, E-return)
Therefore we get the desired from Definition 85

(b) $r - r' \leqslant J$:
From (E:Unit) we know that $\exists p.r = p + K^{unit}$, $r' = p$ and since we know that $J = K^{unit}$, therefore we are done

2. base:

$$\frac{}{\Sigma; . \vdash^{q+K^{base}}_{q} c : \mathsf{b} \rightsquigarrow E_t} \text{unit}$$

where

$E_t = \lambda u.\mathsf{release} - = u \ \mathsf{in} \ \mathsf{bind} - = \uparrow^{K^{base}} \ \mathsf{in} \ \mathsf{bind} \ a = \mathsf{store}(!c) \ \mathsf{in} \ \mathsf{ret}(a)$

$E'_t = \mathsf{release} - = u \ \mathsf{in} \ \mathsf{bind} - = \uparrow^{K^{base}} \ \mathsf{in} \ \mathsf{bind} \ a = \mathsf{store}(!c) \ \mathsf{in} \ \mathsf{ret}(a)$

To prove: $(T, x\delta_{sf}, E_t \ () \ \delta_t\delta_{tf}) \in \lfloor \mathsf{b} \rceil^{V,H}_{\mathcal{E}}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t$ s.t $V, H \vdash^r_{r'} c \Downarrow_t c, H$. From (E:base) we know that $t = 1$

Therefore it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.E_t\ ()\Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T-1, (), {}^t v_f) \in \lfloor \mathsf{b} \rceil_{\mathcal{V}}$:
We choose ${}^t v_t, {}^t v_f, J$ as $E_t', !c, K^{base}$ respectively
Since from E-app we know that $E_t \Downarrow E_t'$, also since $E_t' \Downarrow^{K^{base}} !c$ (from E-release, E-bind, E-store, E-return)
Therefore we get the desired from Definition 85

(b) $r - r' \leqslant J$:
From (E:base) we know that $\exists p.r = p + K^{base}$, $r' = p$ and since we know that $J = K^{base}$, therefore we are done

3. var:

$$\frac{}{\Sigma; x : \tau \vdash_q^{q+K^{var}} x : \tau \rightsquigarrow E_t} \text{var}$$

where

$E_t = \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a)$

$E_t' = \mathsf{release} - = () \text{ in } \mathsf{bind} - = \uparrow^{K^{var}} \text{ in } \mathsf{bind}\, a = \mathsf{store}\, x \text{ in } \mathsf{ret}(a)$

To prove: $(T, x\delta_{sf}, E_t\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t$ s.t $V, H \vdash_{r'}^{r} x \Downarrow_t V(x), H$. From (E:Var) we know that $t = 1$

Therefore it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.E_t\ ()\Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T-1, V(x), {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}$:
We choose ${}^t v_t, {}^t v_f, J$ as $E_t', \delta_t(x)$ respectively
Since from E-app we know that $E_t \Downarrow E_t'$, also since $E_t \Downarrow^{K^{var}} \delta_t(x)$ (from E-release, E-bind, E-store, E-return)
Therefore we get the desired from Definition 85 and Lemma 91

(b) $r - r' \leqslant J$:
From (E:VAR) we know that $\exists p.r = p + K^{var}$, $r' = p$ and $J = K^{var}$, so we are done

4. app:

$$\frac{\tau_1 \xrightarrow{q/q'} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash_{q'-K_2^{app}}^{q+K_1^{app}} f\ x : \tau_2 \rightsquigarrow E_t} \text{app}$$

where

$E_t = \lambda u.E_0$

$E_0 = \mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K_1^{app}} \text{ in } \mathsf{bind}\, P = \mathsf{store}() \text{ in } E_1$

$E_1 = \mathsf{bind}\, f_1 = (f\ P\ x) \text{ in } \mathsf{release}\, f_2 = f_1 \text{ in } \mathsf{bind} - = \uparrow^{K_2^{app}} \text{ in } \mathsf{bind}\, f_3 = \mathsf{store}\, f_2 \text{ in } \mathsf{ret}\, f_3$

To prove: $(T, f\ x, E_t\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau_2 \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t < T$ s.t $V, H \vdash_{r'}^{r} f\ x\delta_{sf} \Downarrow_t {}^s v, H'$

and it suffices to prove that

$\exists^t v_t, {}^t v_f, J.E_t\ ()\Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T-t, {}^s v, {}^t v_f) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}^{H'} \wedge r - r' \leqslant J$ (F-A0)

Since we are given that $(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}}^{H}$ therefore from Definition 86 we know that

$(T, \delta_{sf}(f)\ \delta_{sf}, \delta_{tf}(f)\ \delta_{tf}) \in \lfloor (\tau_1 \xrightarrow{q/q'} \tau_2) \rfloor_{\mathcal{V}'}^{H}$

From Definition 84 we know that $\delta_{sf}(f) = (f(x) = e_s)$ and $\delta_{tf}(f) = \mathsf{fix} f.\lambda u.\lambda x.e_t$ and we have

$\forall^s v', {}^t v', T' < T.(T', {}^s v', {}^t v') \in \lfloor \tau_1 \rfloor_{\mathcal{V}}^{H} \implies (T', e_s, e_t[()/u][{}^t v'/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$ (F-A1)

Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{H}$ therefore we have
$(T, V(x), \delta_t(x)) \in \lfloor \tau_1 \rfloor_{\mathcal{V}}^{H}$

This means from Lemma 87 we also have $(T-1, V(x), \delta_t(x)) \in \lfloor \tau_1 \rfloor_{\mathcal{V}}^{H}$

Instantiating (F-A1) with $T-1, V(x), \delta_t(x)$ we get
$(T-1, e_s, e_t[()/u][\delta_t(x)/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f]) \in \lfloor \tau_2 \rfloor_{\mathcal{E}}^{\{x \mapsto V(x)\}, H}$

This means from Definition 84 we have
$\forall H'_1, {}^s v_1, r_1, r'_1, t' < T-1.\ V, H \vdash_{r'_1}^{r_1} e_s \Downarrow_{t'} {}^s v_1, H'_1 \implies$
$\exists^t v_t, {}^t v_f, J_1.e_t[()/u][\delta_t(x)/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f] \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge (T-1-t', {}^s v_1, {}^t v_f) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}^{H'_1} \wedge r_1 - r'_1 \leqslant J_1$ (F-A2)

Since we know that $V, H \vdash_{r'}^{r} f\ x\delta_{sf} \Downarrow_t {}^s v, H'$ where $t < T$ therefore from (E:FunApp) we know that
$V, H \vdash_{r'+K_2^{app}}^{r-K_1^{app}} e_s \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-A2) with $H', {}^s v, r - K_1^{app}, r' + K_2^{app}, t-1$ we get
$\exists^t v_t, {}^t v_f, J_1.e_t[()/u][\delta_t(x)/x][\mathsf{fix} f.\lambda u.\lambda x.e_t/f] \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T-t, {}^s v, {}^t v_f) \in \lfloor \tau_2 \rfloor_{\mathcal{V}}^{H'} \wedge (r - K_1^{app}) - (r' + K_2^{app}) \leqslant J_1$ (F-A3)

From E-release, E-bind, E-store we know that $J = J_1 + K_1^{app} + K_2^{app}$ therefore we get the desired from (F-A3)

5. nil:

$$\frac{}{\Sigma; \varnothing \vdash_q^{q+K^{nil}} nil : L^{\vec{p}}\tau \rightsquigarrow E_t}\ \text{nil}$$

where
$E_t = \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{nil}} \text{ in } \mathsf{bind}\, a = \mathsf{store}() \text{ in } \mathsf{bind}\, b = \mathsf{store}\langle\langle a, nil\rangle\rangle \text{ in } \mathsf{ret}(b)$
To prove: $(T, nil, E_t\ ()\ \delta_t \delta_{tf}) \in \lfloor L^{\vec{p}}\tau \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some
${}^s v, H', {}^s v, t < T$ s.t $\varnothing, \varnothing \vdash_{p'}^{p} nil \Downarrow_t {}^s v, H'$
From (E:NIL) we know that ${}^s v = NULL$, $H' = H$ and $t = 1$ and it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - 1, nil, {}^t v_f) \in \lfloor L^{\vec{p}} \tau \rfloor_{\mathcal{V}}$:
From E-bind, E-release, E-return we know that ${}^t v = \langle\langle (), nil \rangle\rangle$ therefore from Definition 84 we get the desired

(b) $p - p' \leqslant J$:
Here $p = q + K^{nil}$, $p' = q$ and $J = K^{nil}$, so we are done

6. cons:

$$\frac{\vec{p} = (p_1, \ldots, p_k)}{\Sigma; x_h : \tau, x_t : L^{(\lhd\ \vec{p})} \tau \vdash_q^{q+p_1+K^{cons}} cons(x_h, x_t) : L^p \tau \leadsto E_t} \text{ cons}$$

where

$E_t = \lambda u.\mathsf{release} - = u \text{ in } \mathsf{bind} - = \uparrow^{K^{cons}} \text{ in } E_0$

$E_0 = x_t; x.\, \mathsf{let} \langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_1$

$E_1 = \mathsf{release} - = x_1 \text{ in } \mathsf{bind}\ a = \mathsf{store}() \text{ in } \mathsf{store} \langle\langle a, x_h :: x_2 \rangle\rangle$

$E_t' = \mathsf{release} - = () \text{ in } \mathsf{bind} - = \uparrow^{K^{cons}} \text{ in } E_0$

To prove: $(T, cons(x_h, x_t), E_t\ ()\ \delta_t \delta_{tf}) \in \lfloor L^{\vec{p}} \tau \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some
${}^s v, H', {}^s v, p, p', t < T$ s.t $\emptyset, \emptyset \vdash_{p'}^{p} cons(x_h, x_t)\delta_{sf} \Downarrow_t {}^s v, H'$
and it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.E_t\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, H'(\ell), {}^t v_f) \in \lfloor L^{\vec{p}} \tau \rfloor_{\mathcal{V}}^{H}$:
From (E-app) of λ-Amor we know that $E_t\ () \Downarrow E_t'$
Also from E-release, E-bind, E-store we know that ${}^t v_f = \langle\langle (), \delta_t(x_h) :: \delta_t(x_t) \downarrow_2 \rangle\rangle$

Therefore it suffices to prove that $(T - t, \ell, \langle\langle (), \delta_t(x_h) :: \delta_t(x_t) \downarrow_2 \rangle\rangle) \in \lfloor L^{\vec{p}} \tau \rfloor_{\mathcal{V}}^{H'}$

From Definition 84 it further suffices to prove that
$(T - t, \ell, \delta_t(x_h) :: \delta_t(x_t) \downarrow_2) \in \lfloor L\ \tau \rfloor_{\mathcal{V}}^{H'}$

Since from (E:CONS) rule of univariate RAML we know that $H' = H[\ell \mapsto v]$ where $v = (V(x_h), V(x_t))$

Therefore it further suffices to prove that
$(T - t, V(x_h), \delta_t(x_h)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'}$ and $(T - t, V(x_t), \delta_t(x_t) \downarrow_2) \in \lfloor L\ \tau \rfloor_{\mathcal{V}}^{H'}$

Since we are given that $(T, V, \delta_{tf}) \in \lfloor \Sigma \rfloor_{\mathcal{V}'}^{V,H}$ therefore from Definition 85 and Lemma 87 it means we have
$(T - t, V(x_h), \delta_t(x_h)) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H}$ (F-C1)
and
$(T - t, V(x_t), \delta_t(x_t)) \in \lfloor L^{\lhd \vec{p}} \tau \rfloor_{\mathcal{V}}^{H}$
This means we also have $(T - t, V(x_t), \delta_t(x_t) \downarrow_2) \in \lfloor L\ \tau \rfloor_{\mathcal{V}}^{H}$ (F-C2)

Since $H' = H[\ell \mapsto v]$ where $v = (V(x_h), V(x_t))$ therefore we also have
We get the desired from (F-C1), (F-C2) and Definition 84

(b) $p - p' \leqslant J$:

From (E:CONS) we know that $p = q' + K^{cons}$ and $p' = q'$ for some q'. Also we know that $J = K^{cons}$. Therefore we are done.

7. match:

$$\frac{\vec{p} = (p_1, \ldots, p_k) \qquad \begin{array}{c}\Sigma; \Gamma \vdash^{q - K_1^{matN}}_{q' + K_2^{matN}} e_1 : \tau' \rightsquigarrow e_{a1} \\ \Sigma; \Gamma, h : \tau, t : L^{(\lhd\ \vec{p})}\tau \vdash^{q + p_1 - K_1^{matC}}_{q' + K_2^{matN}} e_2 : \tau' \rightsquigarrow e_{a2}\end{array}}{\Sigma; \Gamma; x : L^{p}\tau \vdash^{q}_{q'} \mathsf{match}\ x\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2 : \tau' \rightsquigarrow \qquad \lambda u.E_0}\ \text{match}$$

where

$E_0 = \mathsf{release} - = u \mathsf{\ in\ } E_{0.1}$

$E_{0.1} = x; a. \mathsf{let}\langle\langle x_1, x_2\rangle\rangle = a \mathsf{\ in\ } E_1$

$E_1 = \mathsf{match}\ x_2\ \mathsf{with}\ |nil \mapsto E_2\ |h :: l_t \mapsto E_3$

$E_2 = \mathsf{bind} - = \uparrow^{K_1^{matN}} \mathsf{\ in\ } E_{2.1}$

$E_{2.1} = \mathsf{bind}\, b = \mathsf{store}() \mathsf{\ in\ } E_2'$

$E_2' = \mathsf{bind}\, c = (e_{a1}\ b) \mathsf{\ in\ } E_{2.1}'$

$E_{2.1}' = \mathsf{release}\, d = c \mathsf{\ in\ } E_{2.2}'$

$E_{2.2}' = \mathsf{bind} - = \uparrow^{K_2^{matN}} \mathsf{\ in\ } E_{2.3}'$

$E_{2.3}' = \mathsf{release} - = x_1 \mathsf{\ in\ store}\, d$

$E_3 = \mathsf{bind} - = \uparrow^{K_1^{matC}} \mathsf{\ in\ } E_{3.1}$

$E_{3.1} = \mathsf{release} - = x_1 \mathsf{\ in\ } E_{3.2}$

$E_{3.2} = \mathsf{bind}\, b = \mathsf{store}() \mathsf{\ in\ } E_{3.3}$

$E_{3.3} = \mathsf{bind}\, t = \mathsf{ret}\langle\langle b, l_t\rangle\rangle \mathsf{\ in\ } E_{3.4}$

$E_{3.4} = \mathsf{bind}\, d = \mathsf{store}() \mathsf{\ in\ } E_{3.5}$

$E_{3.5} = \mathsf{bind}\, f = e_{a2}\ d \mathsf{\ in\ } E_{3.6}$

$E_{3.6} = \mathsf{release}\, g = f \mathsf{\ in\ } E_{3.7}$

$E_{3.7} = \mathsf{bind} - = \uparrow^{K_2^{matC}} \mathsf{\ in\ store}\, g$

To prove: $(T, \mathsf{match}\ x\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2, \lambda u.E_0\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau' \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^{s}v, H', {}^{s}v, p, p', t < T$ s.t $V, H \vdash^{p}_{p'} (\mathsf{match}\ x\ \mathsf{with}\ |nil \mapsto e_1\ |h :: t \mapsto e_2)\delta_{sf} \Downarrow_t {}^{s}v, H'$

2 cases arise:

(a) $V(x) = NULL$:

Since $(T, V, \delta_t) \in \lfloor \Gamma \rfloor_{\mathcal{V}}^{V,H}$ therefore from Definition 85 and Definition 84 we have $\delta_t(x) = \langle\langle (), nil\rangle\rangle$

<u>IH</u>: $(T - 1, e_1\delta_{sf}, e_{a1}\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau' \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have

$\forall H'_1, {}^s v_1, p_1, p'_1, t_1.\ V, H \vdash^{p_1}_{p'_1} e_1 \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists^t v_{t1}, {}^t v_{f1}, J_1.e_{a1} \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T -1 - t_1, {}^s v_1, {}^t v_{f1}) \in \lfloor \tau' \rfloor^{H'_1}_{\mathcal{V}} \wedge p_1 - p'_1 \leqslant J_1$ (F-RUA-M0)

Since we are given that $V, H \vdash^{p}_{p'}$ (match x with $|nil \mapsto e_1\ |h :: t \mapsto e_2)\delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:MatvhN) we know that $V, H \vdash^{p-K_1^{matN}}_{p'+K_2^{matN}} e_1 \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-RUA-M0) with $H', {}^s v, p - K_1^{matN}, p' + K_2^{matN}$ we get
$\exists^t v_{t1}, {}^t v_{f1}, J_1.e_{a1} \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T -t, {}^s v, {}^t v_{f1}) \in \lfloor \tau' \rfloor^{H'}_{\mathcal{V}} \wedge p - K_1^{matN} - p' - K_2^{matN} \leqslant J_1$ (F-RUA-M1)

It suffices to prove that
$\exists^t v_t, {}^t v_f, J.\lambda u.E_0\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in \lfloor \tau' \rfloor^{H'}_{\mathcal{V}} \wedge p - p' \leqslant J$

We choose ${}^t v_t$ as ${}^t v_{t1}$, ${}^t v_f$ as ${}^t v_{f1}$ and J as $J_1 + K_1^{matN} + K_2^{matN}$ and we get the desired from E-bind, E-release, E-store and (F-RUA-M1)

(b) $V(x) = \ell_s$:
Since $(T, V, \delta_t) \in \lfloor \Gamma \rfloor^{V,H}_{\mathcal{V}}$ therefore from Definition 85 and Definition 84 we have
$\delta_t(x) = \langle\langle (), {}^t v_h :: l_t \rangle\rangle$ s.t
$H(\ell_s) = ({}^s v_h, \ell_{ts})$, $({}^s v, {}^t v) \in \lfloor \tau' \rfloor_{\mathcal{V}}$ and $(\ell_s, l_t) \in \lfloor L\ \tau' \rfloor_{\mathcal{V}}$ and

Let $V' = V \cup \{h \mapsto {}^s v_h\} \cup \{t \mapsto \ell_{ts}\}$ and $\delta'_t = \delta_t \cup \{h \mapsto {}^t v_h\} \cup \{t \mapsto l_{ts}\}$
From Definition 85 and Lemma 87 we have $(T -1, V', \delta'_t) \in \lfloor \Gamma, h : \tau, t : L^{\lhd \vec{p}} \tau \rfloor^{V',H}_{\mathcal{V}}$
Therefore from IH we have
$(T -1, e_2\delta_{sf}, e_{a2}\ ()\ \delta'_t\delta_{tf}) \in \lfloor \tau' \rfloor^{V',H}_{\mathcal{E}}$

This means from Definition 84 we have
$\forall H'_2, {}^s v_2, p_2, p'_2, t_1.\ V, H \vdash^{p_1}_{p'_1} e_2 \Downarrow_{t_1} {}^s v_2, H'_2 \implies \exists^t v_{t2}, {}^t v_{f2}, J_2.e_{a2} \Downarrow {}^t v_{t2} \Downarrow^{J_2} {}^t v_{f2} \wedge (T -1 - t_1, {}^s v_2, {}^t v_{f2}) \in \lfloor \tau' \rfloor^{H'_2}_{\mathcal{V}} \wedge p_2 - p'_2 \leqslant J_2$ (F-RUA-M0.0)

Since we are given that $V, H \vdash^{p}_{p'}$ (match x with $|nil \mapsto e_1\ |h :: t \mapsto e_2)\delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:MatvhC) we know that $V, H \vdash^{p-K_1^{matC}}_{p'+K_2^{matC}} e_2 \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-RUA-M0.0) with $H', {}^s v, p - K_1^{matC}, p' + K_2^{matC}, t-1$ we get
$\exists^t v_{t2}, {}^t v_{f2}, J_2.e_{a2} \Downarrow {}^t v_{t2} \Downarrow^{J_2} {}^t v_{f2} \wedge (T -t, {}^s v_2, {}^t v_{f2}) \in \lfloor \tau' \rfloor^{H'_2}_{\mathcal{V}} \wedge p_2 - p'_2 \leqslant J_2$
(F-RUA-M2)

It suffices to prove that
$\exists^t v_t, {}^t v_f, J.\lambda u.E_0\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in \lfloor \tau' \rfloor^{H'}_{\mathcal{V}} \wedge p - p' \leqslant J$

We choose ${}^t v_t$ as ${}^t v_{t2}$, ${}^t v_f$ as ${}^t v_{f2}$ and J as $J_2 + K_1^{matC} + K_2^{matC}$ and we get the desired from E-bind, E-release, E-store and (F-RUA-M2)

8. Share:

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^q_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma; \Gamma, z : \tau \vdash^q_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-unit}$$

$E_0 = \lambda u.E_1$

$E_1 = \mathsf{bind}\, a = coerce_{\mathbf{1},\mathbf{1},\mathbf{1}}\ z\ \mathsf{in}\ \mathsf{let}\langle\langle x, y \rangle\rangle = a\ \mathsf{in}\ e_a\ u$

$coerce_{\mathbf{1},\mathbf{1},\mathbf{1}} \triangleq \lambda u.\, \mathsf{ret}\langle\!\langle !(), !() \rangle\!\rangle$

To prove: $(T, e[z/x, z/y], E_0\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau' \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some
${}^s v, H', {}^s v, p, p', t$ s.t $V, H \vdash^{p}_{p'} e[z/x, z/y]\delta_{sf} \Downarrow_t {}^s v, H'$
And we need to prove
$\exists^t v_t, {}^t v_f, J.E_0\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

Let
$V' = V \cup \{x \mapsto V(z)\} \cup \{y \mapsto V(z)\}$
$\delta'_t = \delta_t \cup \{x \mapsto \delta_t(z)\} \cup \{y \mapsto \delta_t(z)\}$
Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma, z : \mathbf{1} \rceil_{\mathcal{V}}^{V,H}$ therefore from Definition 85 we also have
$(T, V', \delta'_t) \in \lfloor \Gamma, x : \mathbf{1}, y : \mathbf{1} \rceil_{\mathcal{V}}^{V',H}$
<u>IH</u>
$(T, e, e_a\ ()\ \delta'_t\delta_{tf}) \in \lfloor \tau' \rceil_{\mathcal{E}}^{V',H}$
This means from Definition 84 we have
$\forall H'_1, {}^s v_1, p_1, p'_1, t_1.\ V', H \vdash^{p_1}_{p'_1} e \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists^t v_t, {}^t v_f, J.e_a() \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_1, {}^t v_f) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'} \wedge p_1 - p'_1 \leqslant J$
Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^{q}_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = \tau_1 = \tau_2 = \mathsf{b}}{\Sigma; \Gamma, z : \tau \vdash^{q}_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0}\ \text{Share-base}$$

$E_0 = \lambda u.E_1$
$E_1 = \mathsf{bind}\, a = coerce_{\mathsf{b},\mathsf{b},\mathsf{b}}\ z\ \mathsf{in}\ \mathsf{let}\langle\!\langle x, y \rangle\!\rangle = a\ \mathsf{in}\ e_a\ u$

$coerce_{\mathsf{b},\mathsf{b},\mathsf{b}} \triangleq \lambda u.\, \mathsf{let}\, !\, u' = u\ \mathsf{in}\ \mathsf{ret}\langle\!\langle u', u' \rangle\!\rangle$

Similar reasonign as in the unit case above

$$\frac{\begin{array}{c}\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^{q}_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \\ \tau = L^{\vec{p}}\tau'' \qquad \tau_1 = L^{\vec{p_1}}\tau''_1 \qquad \tau_2 = L^{\vec{p_2}}\tau''_2 \qquad \tau'' = \tau''_1 \oplus \tau''_2 \qquad \vec{p} = \vec{p_1} \oplus \vec{p_2}\end{array}}{\Sigma; \Gamma, z : \tau \vdash^{q}_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0}\ \text{Share-list}$$

$E_0 = \lambda u.E_1$
$E_1 = \mathsf{bind}\, a = coerce_{\tau,\tau_1,\tau_2}\ z\ \mathsf{in}\ \mathsf{let}\langle\!\langle x, y \rangle\!\rangle = a\ \mathsf{in}\ e_a\ u$
$coerce_{L^{\vec{p}}\tau, L^{\vec{p_1}}\tau_1, L^{\vec{p_2}}\tau_2} \triangleq \mathsf{fix} f.\lambda\_g.\lambda e.\, \mathsf{let}\, !\, g' = g\ \mathsf{in}\ e; x.\, \mathsf{let}\langle\!\langle p, l \rangle\!\rangle = x\ \mathsf{in}\ E_0$
where

$E_0 \triangleq \mathsf{release} - = p \mathsf{~in~} E_1$

$E_1 \triangleq \mathsf{match~} l \mathsf{~with~} |nil \mapsto E_{2.1} \; |h :: t \mapsto E_3$

$E_{2.1} \triangleq \mathsf{bind~} z_1 = \mathsf{store}() \mathsf{~in~} E_{2.2}$

$E_{2.2} \triangleq \mathsf{bind~} z_2 = \mathsf{store}() \mathsf{~in~} E_{2.3}$

$E_{2.3} \triangleq \mathsf{ret}\langle\langle\langle\langle z_1, nil\rangle\rangle, \langle\langle z_2, nil\rangle\rangle\rangle\rangle$

$E_3 \triangleq \mathsf{bind~} H = g' \; h \mathsf{~in~} E_{3.1}$

$E_{3.1} \triangleq \mathsf{bind~} o_t = () \mathsf{~in~} E_{3.2}$

$E_{3.2} \triangleq \mathsf{bind~} T = f \; g \; \langle\langle o_t, t\rangle\rangle \mathsf{~in~} E_4$

$E_4 \triangleq \mathsf{let}\langle\langle H_1, H_2\rangle\rangle = H \mathsf{~in~} E_5$

$E_5 \triangleq \mathsf{let}\langle\langle T_1, T_2\rangle\rangle = T \mathsf{~in~} E_6$

$E_6 \triangleq T_1; tp_1. \mathsf{let}\langle\langle p'_1, l'_1\rangle\rangle = tp_1 \mathsf{~in~} E_{7.1}$

$E_{7.1} \triangleq T_2; tp_2. \mathsf{let}\langle\langle p'_2, l'_2\rangle\rangle = tp_2 \mathsf{~in~} E_{7.2}$

$E_{7.2} \triangleq \mathsf{release} - = p'_1 \mathsf{~in~} E_{7.3}$

$E_{7.3} \triangleq \mathsf{release} - = p'_2 \mathsf{~in~} E_{7.4}$

$E_{7.4} \triangleq \mathsf{bind~} o_1 = \mathsf{store}() \mathsf{~in~} E_{7.5}$

$E_{7.5} \triangleq \mathsf{bind~} o_2 = \mathsf{store}() \mathsf{~in~} E_8$

$E_8 \triangleq \mathsf{ret}\langle\langle\langle\langle o_1, H_1 :: T_1\rangle\rangle, \langle\langle o_2, H_2 :: T_2\rangle\rangle\rangle\rangle$

To prove: $(T, e[z/x, z/y], E_0 \; () \; \delta_t \delta_{tf}) \in \lfloor \tau' \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some
${}^s v, H', {}^s v, p, p', t < T$ s.t $V, H \vdash^{p}_{p'} e[z/x, z/y]\delta_{sf} \Downarrow_t {}^s v, H'$
And we need to prove
$\exists^t v_t, {}^t v_f, J.E_0 \; () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

Let
$V' = V \cup \{x \mapsto V(z)\} \cup \{y \mapsto V(z)\}$
$\delta'_t = \delta_t \cup \{x \mapsto \delta_t(z)\} \cup \{y \mapsto \delta_t(z)\}$
Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma, z : \tau \rfloor_{\mathcal{V}}^{V,H}$ therefore from Definition 85 we also have $(T, V', \delta'_t) \in \lfloor \Gamma, x : \tau_1, y : \tau_2 \rfloor_{\mathcal{V}}^{V',H}$
<u>IH</u>
$(T, e, e_a \; () \; \delta'_t \delta_{tf}) \in \lfloor \tau' \rfloor_{\mathcal{E}}^{V',H}$
This means from Definition 84 we have
$\forall H'_1, {}^s v_1, p_1, p'_1, t_1. \; V', H \vdash^{p_1}_{p'_1} e \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists^t v_t, {}^t v_f, J.e_a() \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_1, {}^t v_f) \in \lfloor \tau' \rfloor_{\mathcal{V}}^{H'} \wedge p_1 - p'_1 \leqslant J$
Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash^{q}_{q'} e : \tau' \rightsquigarrow e_a \qquad \tau = \tau_1 \curlyvee \tau_2 \qquad \tau = (\tau_a, \tau_b) \qquad \tau_1 = (\tau'_a, \tau'_b) \qquad \tau_2 = (\tau''_a, \tau''_b) \qquad \tau = \tau_1 \oplus \tau_2}{\Sigma; \Gamma, z : \tau \vdash^{q}_{q'} e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-pair}$$

$E_0 = \lambda u.E_1$

$E_1 = \mathsf{bind}\, a = coerce_{(\tau_a,\tau_b),(\tau'_a,\tau'_b),(\tau''_a,\tau''_b)}\ z\ \mathsf{in}\ \mathsf{let}\langle\!\langle x, y\rangle\!\rangle = a\ \mathsf{in}\ e_a\ u$

$coerce_{(\tau_a,\tau_b),(\tau'_a,\tau'_b),(\tau''_a,\tau''_b)} \triangleq \lambda\_g_1.\lambda\_g_2.\lambda p.\ \mathsf{let}\,!\langle\!\langle p_1, p_2\rangle\!\rangle = p\ \mathsf{in}\ E_0$

where

$E_0 \triangleq \mathsf{let}\,!\, g'_1 = g_1\ \mathsf{in}\ E_1$

$E_1 \triangleq \mathsf{let}\,!\, g'_2 = g_2\ \mathsf{in}\ E_2$

$E_2 \triangleq \mathsf{bind}\, P'_1 = g'_1 p_1\ \mathsf{in}\ E_3$

$E_3 \triangleq \mathsf{bind}\, P'_2 = g'_2 p_2\ \mathsf{in}\ E_4$

$E_4 \triangleq \mathsf{let}\,!\langle\!\langle p'_{11}, p'_{12}\rangle\!\rangle = P'_1\ \mathsf{in}\ E_5$

$E_5 \triangleq \mathsf{let}\,!\langle\!\langle p'_{21}, p'_{22}\rangle\!\rangle = P'_2\ \mathsf{in}\ E_6$

$E_6 \triangleq \mathsf{ret}\langle\!\langle p'_{11}, p'_{21}\rangle\!\rangle, \langle\!\langle p'_{12}, p'_{22}\rangle\!\rangle$

Same reasoning as in the list subcase above

9. Sub:

$$\frac{\Sigma; \Gamma \vdash^{q}_{q'} e : \tau \rightsquigarrow e_a \qquad \tau <: \tau'}{\Sigma; \Gamma \vdash^{q}_{q'} e : \tau' \rightsquigarrow e_a}$$

To prove: $(T, e, e_a\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau' \rfloor^{V,H}_{\mathcal{E}}$

<u>IH</u>: $(T, e, e_a\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau \rfloor^{V,H}_{\mathcal{E}}$

We get the desired from IH and Lemma 94

10. Relax:

$$\frac{\Sigma; \Gamma \vdash^{p}_{p'} e : \tau \rightsquigarrow e_a \qquad q \geqslant p \qquad q - p \geqslant q' - p'}{\Sigma; \Gamma \vdash^{q}_{q'} e : \tau \rightsquigarrow E_t}$$

where

$E_t = \lambda o.E_0$

$E_0 = \mathsf{release} - = o\ \mathsf{in}\ E_1$

$E_1 = \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ E_2$

$E_2 = \mathsf{bind}\, b = e_a\ a\ \mathsf{in}\ E_3$

$E_3 = \mathsf{release}\, c = b\ \mathsf{in}\ \mathsf{store}\, c$

To prove: $(T, e, E_t\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau \rfloor^{V,H}_{\mathcal{E}}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t < T$ s.t $\varnothing, \varnothing \vdash^{r}_{r'} e \Downarrow_t {}^s v, H'$

And it suffices to prove that

$\exists^t v_t, {}^t v_f, J.E_t\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}} \wedge r - r' \leqslant J$ (F-R0)

IH: $(T, e, e_a\ ()\ \delta_t \delta_{tf}) \in \lfloor \tau \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have

$\forall^s v_1, H_1', r_1, r_1', t_1 < T . V, H \vdash^{r_1}_{r_1'} e \Downarrow_{t_1} {}^s v_1, H' \implies \exists^t v_t, {}^t v_f, J.e_a\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}} \wedge r - r' \leqslant J$

Instantiating it with the given ${}^s v, H', r, r', t$ we get

$\exists^t v_t', {}^t v_f', J'.e_a\ () \Downarrow {}^t v_t' \Downarrow^{J'} {}^t v_f' \wedge (T - t, {}^s v, {}^t v_f') \in \lfloor \tau \rfloor_{\mathcal{V}} \wedge r - r' \leqslant J'$ (F-R1)

In order to prove (F-R0) we choose ${}^t v_t, {}^t v_f, J$ as ${}^t v_t', {}^t v_f', J'$ and we get the desired from E-app, E-release, E-bind, E-store and (F-R1)

11. Super:

$$\frac{\Sigma; \Gamma, x : \tau_1 \vdash^{q}_{q'} e : \tau \rightsquigarrow e_a \qquad \tau_1' <: \tau_1}{\Sigma; \Gamma, x : \tau_1' \vdash^{q}_{q'} e : \tau \rightsquigarrow e_a} \text{Super}$$

Given: $(T, V, \delta_t) \in \lfloor \Gamma, x : \tau_1' \rfloor_{\mathcal{V}}^{H}$

To prove: $(T, e, e_a\ ()\ \delta_t \delta_{tf}) \in \lfloor \tau \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 it suffices to prove that

$\forall H', {}^s v, p, p', t < T\ .\ V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_a\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

This means given some $H', {}^s v, p, p', t < T$ s.t $V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J.e_a\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (F-Su0)

Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma, x : \tau_1' \rfloor_{\mathcal{V}}^{H}$ therefore from Definition 85 we know that $(T, V(x), \delta_t(x)) \in \lfloor \tau_1' \rfloor_{\mathcal{V}}^{H}$

Therefore from Lemma 93 we know that $(T, V(x), \delta_t(x)) \in \lfloor \tau_1 \rfloor_{\mathcal{V}}^{H}$

IH: $(T, e, e_a\ ()\ \delta_t \delta_{tf}) \in \lfloor \tau \rfloor_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have

$\forall H_i', {}^s v_i, p_i, p_i', t_1.\ V, H \vdash^{p_i}_{p_i'} e \Downarrow_{t_1} {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_a\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_i, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p_i - p_i' \leqslant J$

Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

12. Let:

$$\frac{\Sigma; \Gamma_1 \vdash^{q - K_1^{let}}_{p} e_1 : \tau_1 \rightsquigarrow e_{a1} \qquad \Sigma; \Gamma_2, x : \tau_1 \vdash^{p - K_2^{let}}_{q' + K_3^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma; \Gamma_1, \Gamma_2 \vdash^{q}_{q'} \mathsf{let}\ x = e_1\ in\ e_2 : \tau \rightsquigarrow E_t} \text{Let}$$

where

$E_t = \lambda u.E_0$

$E_0 = \mathsf{release} - = u \mathsf{~in~} E_1$

$E_1 = \mathsf{bind} - = \uparrow^{K_1^{let}} \mathsf{~in~} E_2$

$E_2 = \mathsf{bind}\, a = \mathsf{store}() \mathsf{~in~} E_3$

$E_3 = \mathsf{bind}\, b = e_{a1}\ a \mathsf{~in~} E_4$

$E_4 = \mathsf{release}\, x = b \mathsf{~in~} E_5$

$E_5 = \mathsf{bind} - = \uparrow^{K_2^{let}} \mathsf{~in~} E_6$

$E_6 = \mathsf{bind}\, c = \mathsf{store}() \mathsf{~in~} E_7$

$E_7 = \mathsf{bind}\, d = e_{a2}\ c \mathsf{~in~} E_8$

$E_8 = \mathsf{release}\, f = d \mathsf{~in~} E_9$

$E_9 = \mathsf{bind} - = \uparrow^{K_3^{let}} \mathsf{~in~} E_{10}$

$E_{10} = \mathsf{bind}\, g = \mathsf{store}\, f \mathsf{~in~} \mathsf{ret}\, g$

To prove: $(T, \mathsf{let}\ x = e_1\ in\ e_2, E_t\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t < T$ s.t $V, H \vdash^{r}_{r'} (\mathsf{let}\ x = e_1\ in\ e_2)\delta_{sf} \Downarrow_t {}^s v, H'$

it suffices to prove that

$\exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge r - r' \leqslant J$ (F-L0)

Since we are given that $(T, V, \delta_t)\ in \lfloor \Gamma_1, \Gamma_2 \rceil_{\mathcal{V}}^{H}$ therefore we know that

$\exists V_1, V_2, \delta_t^1, \delta_t^2$ s.t $V = V_1, V_2,\ \delta_t = \delta_t^1, \delta_t^2$ and

$(T, V_1, \delta_t^1) \in \lfloor \Gamma_1 \rceil_{\mathcal{V}}^{H}$ and $(T, V_2, \delta_t^2) \in \lfloor \Gamma_2 \rceil_{\mathcal{V}}^{H}$

<u>IH1</u>

$(T, e_1, e_{a1}\ ()\ \delta_t^1\delta_{tf}) \in \lfloor \tau_1 \rceil_{\mathcal{E}}^{V_1,H}$

This means from Definition 84 we have

$\forall H_1', {}^s v_1, p_1, p_1', t_1.\ V, H \vdash^{p_1}_{p_1'} e_1 \Downarrow_{t_1} {}^s v_1, H' \implies \exists^t v_{t1}, {}^t v_{f1}, J_1.e_{a1}\ () \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T - t_1, {}^s v_1, {}^t v_{f1}) \in \lfloor \tau_1 \rceil_{\mathcal{V}}^{H_1'} \wedge p_1 - p_1' \leqslant J_1$ (F-L1)

Since we know that $V, H \vdash^{r}_{r'} (\mathsf{let}\ x = e_1\ in\ e_2)\delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:Let) we know that $\exists H_1', {}^s v_1, r_1, t_1$ s.t $V, H \vdash^{r - K_1^{let}}_{r_1} e_1\delta_{sf} \Downarrow_{t_1} {}^s v_1, H_1'$

Instantiating (F-L1) with $H_1', {}^s v_1, r - K_1^{let}, r_1, t_1$ we get

$\exists^t v_{t1}, {}^t v_{f1}, J_1.e_{a1}\ () \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T - t_1, {}^s v_1, {}^t v_{f1}) \in \lfloor \tau_1 \rceil_{\mathcal{V}}^{H_1'} \wedge r - K_1^{let} - r_1 \leqslant J_1$ (F-L1.1)

<u>IH2</u>

$(T - t_1, e_2, e_{a2}\ ()\ \delta_t^2 \cup \{x \mapsto {}^t v_{f1}\}\delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V_2 \cup \{x \mapsto {}^s v_1\}, H_1'}$

This means from Definition 84 we have

$\forall H_2', {}^s v_2, p_2, p_2', t_2 < T - t_1.\ V, H \vdash^{p_2}_{p_2'} e_2 \Downarrow_{t_2} {}^s v_2, H' \implies \exists^t v_{t2}, {}^t v_{f2}, J_2 . e_{a2}\ () \Downarrow {}^t v_{t2} \Downarrow^{J_2}$ ${}^t v_{f2} \wedge (T - t_1 - t_2, {}^s v_2, {}^t v_{f2}) \in \lfloor \tau \rceil_{\mathcal{V}}^{H_2'} \wedge p_2 - p_2' \leqslant J_2$ (F-L2)

Since we know that $V, H \vdash^{r}_{r'} (\mathsf{let}\ x = e_1\ in\ e_2)\delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:Let) we know that $\exists H_2', {}^s v_2, t_2 < t - t_1$ s.t $V, H \vdash^{r_1 - K_2^{let}}_{r' + K_3^{let}} e_2 \delta_{sf} \Downarrow_{t_2} {}^s v, H_2'$

Instantiating (F-L2) with $H_2', {}^s v, r_1 - K_2^{let}, r' + K_3^{let}, t_2$ we get

$\exists^t v_{t2}, {}^t v_{f2}, J_2 . e_{a2}\ () \Downarrow {}^t v_{t2} \Downarrow^{J_2} {}^t v_{f2} \wedge (T - t_1 - t_2, {}^s v, {}^t v_{f2}) \in \lfloor \tau \rceil_{\mathcal{V}}^{H_2'} \wedge r_1 - K_2^{let} - (r' + K_3^{let}) \leqslant J_2$ (F-L2.1)

In order to prove (F-L0) we choose ${}^t v_t$ as ${}^t v_{t2}$, ${}^t v_f$ as ${}^t v_{f2}$, J as $J_1 + J_2 + K_1^{let} + K_2^{let} + K_3^{let}$, t as $t_1 + t_2 + 1$ and we get the desired from (F-L1.1) and (F-L2.1) and Lemma 87

13. Pair:

$$\frac{}{\Sigma; x_1 : \tau_1, x_2 : \tau_2 \vdash^{q + K^{pair}}_{q} (x_1, x_2) : (\tau_1, \tau_2) \rightsquigarrow E_t}\ \text{pair}$$

where

$E_t = \lambda u . E_0$

$E_0 = \mathsf{release} - = u \ \mathsf{in}\ E_1$

$E_1 = \mathsf{bind} - = \uparrow^{K^{pair}} \mathsf{in}\ E_2$

$E_2 = \mathsf{bind}\ a = \mathsf{store}(x_1, x_2)\ \mathsf{in}\ \mathsf{ret}\ a$

Given: $(T, V, \delta_t) \in \lfloor x_1 : \tau_1, x_2 : \tau_2 \rceil_{\mathcal{V}}^{H}$
To prove: $(T, (x_1, x_2), E_t\ ()\ \delta_t \delta_{tf}) \in \lfloor (\tau_1, \tau_2) \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 it suffices to prove that

$\forall H', {}^s v, r, r', t < T\ .\ V, H \vdash^{r}_{r'} (x_1, x_2) \Downarrow_t {}^s v, H' \implies \exists^t v_t, {}^t v_f, J . E_t\ () \Downarrow {}^t v_t \Downarrow^{J} {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor (\tau_1, \tau_2) \rceil_{\mathcal{V}}^{H'} \wedge r - r' \leqslant J$

This means given some $H', {}^s v, r, r', t < T$ s.t $V, H \vdash^{r}_{r'} (x_1, x_2) \Downarrow_t {}^s v, H'$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J . E_t\ () \Downarrow {}^t v_t \Downarrow^{J} {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor (\tau_1, \tau_2) \rceil_{\mathcal{V}}^{H'} \wedge r - r' \leqslant J$ (F-P0)

This means we need to prove that $\exists^t v_t, {}^t v_f, J$

- $E_t\ () \Downarrow {}^t v_t \Downarrow^{J} {}^t v_f$:
  From E-app, E-release,E-bind,E-tick, E-store and E-return we know that ${}^t v_t = E_0$, ${}^t v_f = (\delta_t(x_1), \delta_t(x_2))$ and $J = K^{pair}$
- $(T - t, {}^s v, {}^t v_f) \in \lfloor (\tau_1, \tau_2) \rceil_{\mathcal{V}}^{H'}$:
  Since we are given that $V, H \vdash^{r}_{r'} (x_1, x_2) \Downarrow_t {}^s v, H'$, therefore from (E:Pair) we know that ${}^s v = \ell$ where $\ell \notin dom(H)$ and $H' = H[\ell \mapsto (V(x_1), V(x_2))]$

  Since we are given that $(T, V, \delta_t) \in \lfloor x_1 : \tau_1, x_2 : \tau_2 \rceil_{\mathcal{V}}^{H}$ therefore from Definition 85, Definition 84 and Lemma 87 we get the desired.

- $r - r' \leqslant J$:
  From (E:Pair) we know that $\exists p.r = p + K^{pair}$ and $r' = p$. Since we know that $J = K^{pair}$, therefore we are done.

14. MatP:

$$\frac{\tau = (\tau_1, \tau_2) \qquad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash^{q-K_1^{matP}}_{q'+K_2^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash^q_q \mathit{match}\ x\ \mathit{with}\ (x_1, x_2) \to e : \tau' \rightsquigarrow E_t} \text{ matP}$$

where

$E_t = \lambda u.E_0$

$E_0 = \mathsf{release} - = u \mathsf{\ in\ } E_1$

$E_1 = \mathsf{bind} - = \uparrow^{K_1^{matP}} \mathsf{\ in\ } E_2$

$E_2 = \mathsf{let}\langle\langle x_1, x_2 \rangle\rangle = x \mathsf{\ in\ } E_3$

$E_3 = \mathsf{bind}\, a = \mathsf{store}() \mathsf{\ in\ } E_4$

$E_4 = \mathsf{bind}\, b = e_t\ a \mathsf{\ in\ } E_5$

$E_5 = \mathsf{release}\, c = b \mathsf{\ in\ } E_6$

$E_6 = \mathsf{bind} - = \uparrow^{K_2^{matP}} \mathsf{\ in\ } E_7$

$E_7 = \mathsf{bind}\, d = \mathsf{store}\, c \mathsf{\ in\ ret}\, d$

Given: $(T, V, \delta_t) \in \lfloor \Gamma, x : \tau \rceil_{\mathcal{V}}^H$

To prove: $(T, (\mathit{match}\ x\ \mathit{with}\ (x_1, x_2) \to e), E_t\ ()\ \delta_t\delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 it suffices to prove that

$\forall H', {}^s v, p, p', t < T\ .\ V, H \vdash^p_{p'} (\mathit{match}\, x\, \mathit{with}\, (x_1, x_2) \to e) \Downarrow_t {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.E_t\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

This means given some $H', {}^s v, p, p', t < T$ s.t $V, H \vdash^p_{p'} (\mathit{match}\, x\, \mathit{with}\, (x_1, x_2) \to e) \Downarrow_t {}^s v, H'$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J.E_t\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (F-MP0)

Since we are given that $(T, V, \delta_t) \in \lfloor \Gamma, x : \tau \rceil_{\mathcal{V}}^H$ therefore from Definition 85 and since $\tau = (\tau_1, \tau_2)$ therefore we know that $(T, V(x), \delta_t(x)) \in \lfloor (\tau_1, \tau_2) \rceil_{\mathcal{V}}^H$

This means from Definition 84 that $\exists \ell$ s.t $H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in \lfloor \tau_1 \rceil_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in \lfloor \tau_2 \rceil_{\mathcal{V}}$

<u>IH</u>: $(T, e, e_t\ ()\ \delta_t \cup \{x_1 \mapsto {}^t v_1\} \cup \{x_2 \mapsto {}^t v_2\}\delta_{tf}) \in \lfloor \tau' \rceil_{\mathcal{E}}^{V \cup \{x_1 \mapsto {}^s v_1\} \cup \{x_2 \mapsto {}^s v_2\}, H}$

This means from Definition 84 we have

$\forall H'_i, {}^s v_i, p_i, p'_i, t_1 < T - 1.\ V, H \vdash^{p_i}_{p'_i} e \Downarrow_{t_1} {}^s v, H'_i \implies \exists^t v_{t1}, {}^t v_{f1}, J_1.e\ () \Downarrow {}^t v_{t1} \Downarrow^J {}^t v_{f1} \wedge (T - t_1, {}^s v_i, {}^t v_{f1}) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'_i} \wedge p_i - p'_i \leqslant J_1$

Since we are given that $V, H \vdash^p_{p'} (\mathit{match}\ x\ \mathit{with}\ (x_1, x_2) \to e) \Downarrow {}^s v, H'$ therefore from (E:MatP) we know that

$V \cup \{x_1 \mapsto {}^s v_1\} \cup \{x_2 \mapsto {}^s v_2\}, H \vdash^{p-K_1^{matP}}_{p'+K_2^{matP}} e \Downarrow_{t-1} {}^s v, H'$

Instantiating it with the given $H', {}^s v, p - K_1^{matP}, p' + K_2^{matP}, t-1$ we get

$\exists^t v_{t1}, {}^t v_{f1}, J_1.e\ () \Downarrow {}^t v_{t1} \Downarrow^J {}^t v_{f1} \wedge (T - t, {}^s v, {}^t v_{f1}) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'} \wedge p - K_1^{matP} - (p' + K_2^{matP}) \leqslant J_1$ (F-MP1)

In order to prove (F-MP0) we choose ${}^t v_t$ as ${}^t v_{t1}$, ${}^t v_f$ as ${}^t v_{f1}$, J as $J_1 + K_1^{matP} + K_2^{matP}$ and t_1 as $t-1$ and it suffices to prove that

- $E_t\ () \Downarrow {}^t v_t \Downarrow^J {}^t v_f$:
  We get the desired from E-app, E-bind, E-release, E-store, E-tick, E-return and (F-MP1)
- $(T - t, {}^s v, {}^t v_f) \in \lfloor \tau' \rceil_{\mathcal{V}}^{H'}$:
  From (F-MP1)
- $p - p' \leqslant J$:
  We get this directly from (F-MP1)

15. Augment:

$$\frac{\Sigma; \Gamma \vdash^q_{q'} e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash^q_{q'} e : \tau \rightsquigarrow e_a} \text{ Augment}$$

Given: $(T, V \cup \{x \mapsto {}^s v_x\}, \delta_t \cup \{x \mapsto {}^t v_x\}) \in \lfloor \Gamma, x : \tau' \rceil_{\mathcal{V}}^H$

To prove: $(T, e, e_a\ ()\ \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V \cup \{x \mapsto {}^s v_x\}, H}$

This means from Definition 84 it suffices to prove that

$\forall H', {}^s v, p, p', t < T\ .\ V \cup \{x \mapsto {}^s v_x\}, H \vdash^p_{p'} e_s \Downarrow_t {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_a\ () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

This means given some $H', {}^s v, p, p', t < T$ s.t $V \cup \{x \mapsto {}^s v_x\}, H \vdash^p_{p'} e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J.e_a\ () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (F-Ag0)

Since we are given that $(T, V \cup \{x \mapsto {}^s v_x\}, \delta_t \cup \{x \mapsto {}^t v_x\}) \in \lfloor \Gamma, x : \tau' \rceil_{\mathcal{V}}^H$

therefore from Definition 85 we know that

$(T, V, \delta_t) \in \lfloor \Gamma \rceil_{\mathcal{V}}^H$

<u>IH</u>: $(T, e, e_a\ ()\ \delta_t \delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have

$\forall H'_i, {}^s v_i, p_i, p'_i, t_1 < T\ .\ V, H \vdash^{p_i}_{p'_i} e \Downarrow_{t_1} {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_a\ () \delta_t \delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_i, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge p_i - p'_i \leqslant J$ (F-Ag1)

Since we are given $V \cup \{x \mapsto {}^s v_x\}, H \vdash^p_{p'} e_s \Downarrow_t {}^s v, H'$ and since $x \notin free(e)$ therefore we also have

$V, H \vdash^{p}_{p'} e_s \Downarrow_t {}^s v, H'$

Instantiating (F-Ag1) with the given $H', {}^s v, p, p', t$ we get

$\exists^t v_t, {}^t v_f, J.e_a\ ()\delta_t\delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v_i, {}^t v_f) \in \lfloor \tau \rceil^{H'}_{\mathcal{V}} \wedge p_i - p'_i \leqslant J$

Also since $x \notin free(e)$ therefore we get

$\exists^t v_t, {}^t v_f, J.e_a\ ()\delta_t \cup \{x \mapsto {}^t v_x\}\delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v_i, {}^t v_f) \in \lfloor \tau \rceil^{H'}_{\mathcal{V}} \wedge p_i - p'_i \leqslant J$

$\square$

Lemma 93 (Value subtyping lemma)**.** $\forall \tau, \tau', H, {}^s v, {}^t v, T.$
$\tau <: \tau' \wedge (T, {}^s v, {}^t v) \in \lfloor \tau \rceil^H_{\mathcal{V}} \implies (T, {}^s v, {}^t v) \in \lfloor \tau' \rceil^H_{\mathcal{V}}$

Proof. Proof by induction on the subtyping relation of Univariate RAML

1. Unit:

$$\frac{}{unit <: unit}$$

Given: $(T, {}^s v, {}^t v) \in \lfloor unit \rceil^H_{\mathcal{V}}$

To prove: $(T, {}^s v, {}^t v) \in \lfloor unit \rceil^H_{\mathcal{V}}$

Trivial

2. Base:

$$\frac{}{\mathsf{b} <: \mathsf{b}}$$

Given: $(T, {}^s v, {}^t v) \in \lfloor \mathsf{b} \rceil^H_{\mathcal{V}}$

To prove: $(T, {}^s v, {}^t v) \in \lfloor \mathsf{b} \rceil^H_{\mathcal{V}}$

Trivial

3. Pair:

$$\frac{\tau_1 <: \tau'_1 \qquad \tau_2 <: \tau'_2}{(\tau_1, \tau_2) <: (\tau'_1, \tau'_2)}$$

Given: $(T, {}^s v, {}^t v) \in \lfloor (\tau_1, \tau_2) \rceil^H_{\mathcal{V}}$

To prove: $(T, {}^s v, {}^t v) \in \lfloor (\tau'_1, \tau'_2) \rceil^H_{\mathcal{V}}$

From Definition 84 we know that ${}^s v = \ell$ s.t

$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in \lfloor \tau_1 \rceil_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in \lfloor \tau_2 \rceil_{\mathcal{V}}$ (S-P0)

<u>IH1</u> $(T, {}^s v_1, {}^t v_1) \in \lfloor \tau'_1 \rceil^H_{\mathcal{V}}$

<u>IH2</u> $(T, {}^s v_2, {}^t v_2) \in \lfloor \tau'_2 \rceil^H_{\mathcal{V}}$

Again from Definition 84 it suffices to prove that

$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in \lfloor \tau'_1 \rceil_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in \lfloor \tau'_2 \rceil_{\mathcal{V}}$

We get this directly from (S-P0), IH1 and IH2

4. List:

$$\frac{\tau_1 <: \tau_2 \qquad \vec{p} \geqslant \vec{q}}{L^{\vec{p}}\tau_1 <: L^{\vec{q}}\tau_2}$$

Given: $(T, {}^s v, {}^t v) \in \lfloor L^{\vec{p}}\tau_1 \rfloor_{\mathcal{V}}^{H}$
To prove: $(T, {}^s v, {}^t v) \in \lfloor L^{\vec{q}}\tau_2 \rfloor_{\mathcal{V}}^{H}$

From Definition 84 we know that ${}^s v = l_s$ and ${}^t v = \langle\langle (), l_t \rangle\rangle$ s.t $(T, l_s, l_t) \in \lfloor L\ \tau_1 \rfloor_{\mathcal{V}}$
Similarly from Definition 84 it suffices to show that
$(T, l_s, l_t) \in \lfloor L\ \tau_2 \rfloor_{\mathcal{V}}$

We induct on $(T, l_s, l_t) \in \lfloor L\ \tau_1 \rfloor_{\mathcal{V}}$

- Base case:
  In this case $l_s = NULL$ and $l_t = nil$:
  It suffices to prove that $(T, NULL, nil) \in \lfloor L\ \tau_2 \rfloor_{\mathcal{V}}$
  This holds trivially from Definition 84
- Inductive case
  In this case we have $l_s = \ell$ and $l_t = {}^t v_h :: l_{tt}$:
  It suffices to prove that $(T, \ell, {}^t v_h :: l_{tt}) \in \lfloor L\ \tau_2 \rfloor_{\mathcal{V}}$
  Again from Definition 84 it suffices to show that
  $\exists^s v_{h1}, \ell_{s1}.H(\ell) = ({}^s v_{h1}, \ell_{s1}) \wedge (T, {}^s v_{h1}, {}^t v_h) \in \lfloor \tau_2 \rfloor_{\mathcal{V}} \wedge (T, \ell_{s1}, l_{tt}) \in \lfloor L\ \tau_2 \rfloor_{\mathcal{V}}$

  Since we are given that $(T, \ell, {}^t v_h :: l_{tt}) \in \lfloor L\ \tau_1 \rfloor_{\mathcal{V}}$ therefore from Definition 84 we have
  $\exists^s v_h, \ell_s.H(\ell) = ({}^s v_h, \ell_s) \wedge (T, {}^s v_h, {}^t v_h) \in \lfloor \tau_1 \rfloor_{\mathcal{V}} \wedge (T, \ell_s, l_{tt}) \in \lfloor L\ \tau_1 \rfloor_{\mathcal{V}}$ (S-L1)

  We choose ${}^s v_{h1}$ as ${}^s v_h$ and ℓ_{s1} as ℓ_s

  – $H(\ell) = ({}^s v_h, \ell_s)$:
    Directly from (S-L1)
  – $(T, {}^s v_h, {}^t v_h) \in \lfloor \tau_1 \rfloor_{\mathcal{V}}$:
    From IH of outer induction
  – $(T, \ell_s, l_{tt}) \in \lfloor L\ \tau_2 \rfloor_{\mathcal{V}}$:
    From IH of inner induction

□

Lemma 94 (Expression subtyping lemma). $\forall \tau, \tau', V, H, e_s, e_t.$
$\tau <: \tau' \wedge (T, e_s, e_t) \in \lfloor \tau \rfloor_{\mathcal{E}}^{V,H} \implies (T, e_s, e_t) \in \lfloor \tau' \rfloor_{\mathcal{E}}^{V,H}$

Proof. From Definition 84 we are given that
$\forall H', {}^s v, p, p', t < T\ .\ V, H \vdash_{p'}^{p} e_s \Downarrow_t {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (SE0)

Also from Definition 84 it suffices to prove that
$\forall H', {}^s v, p, p', t_1 < T\ .\ V, H \vdash_{p'}^{p} e_s \Downarrow_{t_1} {}^s v, H' \implies \exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in \lfloor \tau' \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

This means given some $H', {}^s v, p, p', t_1 < T$ s.t $V, H \vdash_{p'}^{p} e_s \Downarrow_{t_1} {}^s v, H'$ it suffices to prove that

$\exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in \lfloor \tau' \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$

We instantiate (SE0) with $H', {}^s v, p, p', t_1$ and we get
$\exists^t v_t, {}^t v_f, J.e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in \lfloor \tau \rfloor_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (SE1)

We get the desired from (SE1) and Lemma 93

□

2.5.3 Re-deriving Univariate RAML's soundness

Definition 95 (Translation of Univariate RAML stack)**.** $\overline{(V : \Gamma)_H} \triangleq \forall x \in dom(\Gamma).\overline{(V(x))_{H,\Gamma(x)}}$

Definition 96 (Translation of Univariate RAML values)**.**

$$\overline{({}^s v)_{H,\tau}} \triangleq \begin{cases} {}^s v & \tau = unit \\ !^s v & \tau = \mathsf{b} \\ \langle\!\langle (), \overline{({}^s v)_{H,L\ \tau'}} \rangle\!\rangle & \tau = L^{-}\tau' \\ nil & \tau = L\ \tau' \wedge {}^s v = NULL \\ \overline{(H(\ell)\downarrow_1)_{H,\tau'}} :: \overline{(H(\ell)\downarrow_2)_{H,L\tau'}} & \tau = L\ \tau' \wedge {}^s v = \ell \\ \langle\!\langle \overline{(H(\ell)\downarrow_1)_{H,\tau_1}}, \overline{(H(\ell)\downarrow_2)_{H,\tau_2}} \rangle\!\rangle & \tau = (\tau_1, \tau_2) \wedge {}^s v = \ell \end{cases}$$

Lemma 97 (Irrelevance of T for translated value)**.** $\forall^s v, \tau, H.$
$H \models {}^s v \in [\![\tau]\!]$ in RAML $\implies \forall\, T\, .(\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H,\tau}}) \in [\![(\!|\tau|\!)]\!]$ in λ-Amor

Proof. By induction on τ

1. $\tau = unit$:

   To prove: $\forall\, T\, .(\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H,\tau}}) \in [\![(\!|unit|\!)]\!]$

   This means given some T it suffices to prove that
   $(\Phi_H({}^s v : unit), T, \overline{({}^s v)_{H,unit}}) \in [\![\mathbf{1}]\!]$

   We know that $\Phi_H({}^s v : unit) = 0$ therefore it suffices to prove that
   $(0, T, {}^s v) \in [\![\mathbf{1}]\!]$

   Since we know that ${}^s v \in [\![unit]\!]$ therefore we know that ${}^s v = ()$
   Therefore we get the desired directly from Definition 66

2. $\tau = \mathsf{b}$:

   To prove: $\forall\, T\, .(\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H,\tau}}) \in [\![(\!|\mathsf{b}|\!)]\!]$

   This means given some T it suffices to prove that
   $(\Phi_H({}^s v : \mathsf{b}), T, \overline{({}^s v)_{H,\tau}}) \in [\![!\mathsf{b}]\!]$

   We know that $\Phi_H({}^s v : \mathsf{b}) = 0$ therefore it suffices to prove that
   $(0, T, !^s v) \in [\![!\mathsf{b}]\!]$
   From Definition 66 it suffices to prove that
   $(0, T, {}^s v) \in [\![\mathsf{b}]\!]$

   Since we know that ${}^s v \in [\![\mathsf{b}]\!]$
   Therefore we get the desired directly from Definition 66

3. $\tau = L^{\vec{q}}\tau'$:

   By induction on ${}^s v$

   - ${}^s v = NULL = []$:
     To prove: $\forall\, T\, .(\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H,L^{\vec{q}}\tau'}}) \in [\![(\!| L^{\vec{q}}\tau' |\!)]\!]$

     This means given some T it suffices to prove that
     $(\Phi_H([] : L^{\vec{q}}\tau'), T, \langle\!\langle (), nil \rangle\!\rangle) \in [\![\exists s.([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau|\!))]\!]$

     We know that $\Phi_H([] : L^{\vec{q}}\tau') = 0$ therefore it suffices to prove that
     $(0, T, \langle\!\langle (), nil \rangle\!\rangle) \in [\![\exists s.([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau|\!))]\!]$
     From Definition 66 it suffices to prove that
     $\exists s'.(0, T, \langle\!\langle (), nil \rangle\!\rangle) \in [\![([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau|\!))[s'/s]]\!]$

     We choose s' as 0 and it suffices to prove that
     $(0, T, \langle\!\langle (), nil \rangle\!\rangle) \in [\![([\phi(\vec{q}, 0)]\, \mathbf{1} \otimes L[0] (\!|\tau|\!))]\!]$

     From Definition 66 it further suffices to prove that
     $\exists p_1, p_2 . p_1 + p_2 \leqslant 0 \wedge (p_1, T, ()) \in [\![([\phi(\vec{q}, 0)]\, \mathbf{1}]\!] \wedge (p_1, T, nil) \in [\![L[0] (\!|\tau|\!))]\!]$
     We choose p_1 and p_2 as 0 and we get the desired directly from Definition 66

   - ${}^s v = \ell = [{}^s v_1, \ldots, {}^s v_n]$:
     To prove: $\forall\, T\, .(\Phi_H([{}^s v_1 \ldots {}^s v_n] : L^{\vec{q}}\tau'), T, \overline{({}^s v)_{H,\tau}}) \in [\![\exists s.([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))]\!]$

     This means given some T it suffices to prove that
     $(\Phi_H([{}^s v_1 \ldots {}^s v_n] : L^{\vec{q}}\tau'), T, \overline{({}^s v)_{H,\tau}}) \in [\![\exists s.([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))]\!]$

     We know that $\Phi_H([{}^s v_1 \ldots {}^s v_n] : L^{\vec{q}}\tau') = (\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau'))$ therefore it suffices to prove that
     $((\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H,\tau}}) \in [\![\exists s.([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))]\!]$
     From Definition 66 it suffices to prove that
     $\exists s'.((\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H,\tau}}) \in [\![([\phi(\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))[s'/s]]\!]$

     We choose s' as n and it suffices to prove that
     $((\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H,\tau}}) \in [\![([\phi(\vec{q}, n)]\, \mathbf{1} \otimes L^n (\!|\tau'|\!))]\!]$

     From Definition 96 we know that $\overline{({}^s v)_{H,\tau}} = \langle\!\langle (), \overline{(H(\ell) \downarrow_1)_{H,\tau'}} :: \overline{(H(\ell) \downarrow_2)_{H,L\tau'}} \rangle\!\rangle$
     From Definition 66 it further suffices to prove that
     $\exists p_1, p_2 . p_1 + p_2 \leqslant (\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')) \wedge (p_1, T, ()) \in [\![[\phi(\vec{q}, n)]\, \mathbf{1}]\!] \wedge (p_2, T$
     $, \overline{(H(\ell) \downarrow_1)_{H,\tau'}} :: \overline{(H(\ell) \downarrow_2)_{H,L\tau'}}) \in [\![L^n (\!|\tau'|\!))]\!] \qquad$ (L0)

     <u>IH</u>
     $(\Phi_H([{}^s v_2 \ldots {}^s v_n] : L^{\lhd\vec{q}}\tau'), T, \overline{(H(\ell) \downarrow_2)_{H,L^{\lhd\vec{q}}\tau'}}) \in [\![\exists s.([\phi(\lhd\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))]\!]$

     We know that $\Phi_H([{}^s v_2 \ldots {}^s v_n] : L^{\lhd\vec{q}}\tau') = (\Phi(n-1, \lhd\vec{q}) + \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau'))$ this means we have
     $((\Phi(n-1, \lhd\vec{q}) + \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{(H(\ell) \downarrow_2)_{H,L^{\lhd\vec{q}}\tau'}}) \in [\![\exists s.([\phi(\lhd\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))]\!]$
     From Definition 66 this means we have
     $\exists s'.((\Phi(n-1, \lhd\vec{q}) + \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{(H(\ell) \downarrow_2)_{H,L^{\lhd\vec{q}}\tau'}}) \in [\![([\phi(\lhd\vec{q}, s)]\, \mathbf{1} \otimes L^s (\!|\tau'|\!))[s'/s]]\!]$

     We know that s' as $n-1$ and we have

$((\Phi(n-1, \lhd\vec{q}) + \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')), T, \overline{(H(\ell) \downarrow_2)_{H, L^{\lhd\vec{q}}\tau'}}) \in [\![([\phi(\lhd\vec{q}, n-1)]\, \mathbf{1} \otimes L^{n-1}(\!|\tau'|\!))]\!]$

From Definition 96 we know that $\overline{(H(\ell) \downarrow_2)_{H, L^{\lhd\vec{q}}\tau'}} = \langle\!\langle (), l_t \rangle\!\rangle$

This means from Definition 66 we have
$\exists p'_1, p'_2. p'_1 + p'_2 \leqslant (\Phi(n-1, \lhd\vec{q}) + \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')) \wedge (p'_1, T, ()) \in [\![[\phi(\lhd\vec{q}, n)]\, \mathbf{1}]\!] \wedge (p'_2, T, l_t) \in [\![L^{n-1}(\!|\tau'|\!)]\!]$ (L1)

Inorder to prove (L0) we choose p_1 as $p'_1 + q_1$ and p_2 as $p'_2 + \Phi_H({}^s v_1 : \tau')$

- $p_1 + p_2 \leqslant (\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau'))$:
  It suffices to prove that
  $p'_1 + q_1 + p'_2 + \Phi_H({}^s v_1 : \tau') \leqslant (\Phi(n, \vec{q}) + \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau'))$

  Since from (L1) we know that $p'_1 \leqslant \Phi(n-1, \lhd\vec{q})$ therefore we also know that
  $p'_1 + q_1 \leqslant \Phi(n, \vec{q})$ (L2)

  Similarly since from (L1) we know that $p'_2 \leqslant \sum_{2 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')$
  Therefore we also have
  $p'_2 + \Phi_H({}^s v_1 : \tau') \leqslant \sum_{1 \leqslant i \leqslant n} \Phi_H({}^s v_i : \tau')$ (L3)

  Combining (L2) and (L3) we get the desired
- $(p_1, T, ()) \in [\![[\phi(\vec{q}, n)]\, \mathbf{1}]\!]$:
  It suffices to prove that $(p'_1 + q_1, T, ()) \in [\![[\phi(\vec{q}, n)]\, \mathbf{1}]\!]$

  Since from (L1) we are given that
  $(p'_1, T, ()) \in [\![[\phi(\lhd\vec{q}, n)]\, \mathbf{1}]\!]$
  Therefore we also have
  $(p'_1 + q_1, T, ()) \in [\![[\phi(\vec{q}, n)]\, \mathbf{1}]\!]$
- $(p_2, T, \overline{(H(\ell) \downarrow_1)_{H,\tau'}} :: \overline{(H(\ell) \downarrow_2)_{H, L\tau'}}) \in [\![L^n(\!|\tau'|\!)]\!]$:
  It suffices to prove that
  $(p'_2 + \Phi_H({}^s v_1 : \tau'), T, \overline{(H(\ell) \downarrow_1)_{H,\tau'}} :: \overline{(H(\ell) \downarrow_2)_{H, L\tau'}}) \in [\![L^n(\!|\tau'|\!)]\!]$

  From Definition 66 it suffices to show that
  $\exists p''_1, p''_2. p''_1 + p''_2 \leqslant \Phi_H({}^s v_1 : \tau') + p'_2 \wedge (p''_1, T, \overline{(H(\ell) \downarrow_1)_{H,\tau'}}) \in [\![\tau']\!] \wedge (p''_2, T, \overline{(H(\ell) \downarrow_2)_{H, L\tau'}}) \in [\![L^{n-1}\tau']\!]$

  We choose p''_1 as $\Phi_H({}^s v_1 : \tau')$ and p''_2 as p'_2 and it suffices to prove that
  * $(p''_1, T, \overline{(H(\ell) \downarrow_1)_{H,\tau'}}) \in [\![\tau']\!]$:
    This means we need to prove that
    $(\Phi_H({}^s v_1 : \tau'), T, \overline{(H(\ell) \downarrow_1)_{H,\tau'}}) \in [\![\tau']\!]$

    We get this from IH of outer induction
  * $(p''_2, T, \overline{(H(\ell) \downarrow_2)_{H, L\tau'}}) \in [\![L^{n-1}\tau']\!]$:
    This means we need to prove that
    $(p'_2, T, \overline{(H(\ell) \downarrow_2)_{H, L\tau'}}) \in [\![L^{n-1}\tau']\!]$

    Since we know that $\overline{(H(\ell) \downarrow_2)_{H, L\tau'}} = l_t$ therefore we get the desired from (L1)

4. $\tau = (\tau_1, \tau_2)$:
   To prove: $\forall\, T\, .(\Phi_H(({}^s v_1, {}^s v_2) : (\tau_1, \tau_2)), T, \overline{({}^s v_1, {}^s v_2)_{H, (\tau_1, \tau_2)}}) \in [\![(\!|(\tau_1, \tau_2)|\!)]\!]$

   This means given some T it suffices to prove that
   $(\Phi_H(({}^s v_1, {}^s v_2) : (\tau_1, \tau_2)), T, \overline{({}^s v_1, {}^s v_2)_{H, (\tau_1, \tau_2)}}) \in [\![(\!|\tau_1|\!) \otimes (\!|\tau_2|\!)]\!]$

We know that $\Phi_H(({}^s v_1, {}^s v_2) : (\tau_1, \tau_2)) = \Phi_H({}^s v_1 : \tau_1) + \Phi_H({}^s v_2 : \tau_2)$ therefore it suffices to prove that

$(\Phi_H({}^s v_1 : \tau_1) + \Phi_H({}^s v_2 : \tau_2), T, (\overline{(H(\ell) \downarrow_1)_{H,\tau_1}}, \overline{(H(\ell) \downarrow_2)_{H,\tau_2}})) \in [\![(\!|\tau_1|\!) \otimes (\!|\tau_2|\!)]\!]$

From Definition 66 it suffices to prove that

$\exists p_1, p_2 . p_1 + p_2 \leqslant (\Phi_H({}^s v_1 : \tau_1) + \Phi_H({}^s v_2 : \tau_2)) \wedge (p_1, T, \overline{(H(\ell) \downarrow_1)_{H,\tau_1}}) \in [\![(\!|\tau_1|\!)]\!] \wedge (p_2, T, \overline{(H(\ell) \downarrow_2)_{H,\tau_2}}) \in [\![(\!|\tau_2|\!)]\!]$

Choosing p_1 as $\Phi_H({}^s v_1 : \tau_1)$ and p_2 as $\Phi_H({}^s v_2 : \tau_2)$ and it suffices to prove that

$(\Phi_H({}^s v_1 : \tau_1), T, \overline{(H(\ell) \downarrow_1)_{H,\tau_1}}) \in [\![(\!|\tau_1|\!)]\!] \wedge (\Phi_H({}^s v_2 : \tau_2), T, \overline{(H(\ell) \downarrow_2)_{H,\tau_2}}) \in [\![(\!|\tau_2|\!)]\!]$

We get this directly from IH1 and IH2

$\square$

Lemma 98 (Irrelevance of T for translated Γ)**.** $\forall {}^s v, \tau, H.$
$H \models V : \Gamma$ in RAML $\implies \forall\, T\, .(\Phi_{V,H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in [\![(\!|\Gamma|\!)]\!]$ in λ-Amor

Proof. To prove: $\forall\, T\, .(\Phi_{V,H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in [\![(\!|\Gamma|\!)]\!]$

This means given soem T it suffices to prove that
$(\Phi_{V,H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in [\![(\!|\Gamma|\!)]\!]$

From Definition 67 it suffices to prove that
$\exists f : \mathcal{V}ars \to \mathcal{P}ots . (\forall x \in dom((\!|\Gamma|\!)). (f(x), T, \overline{(V : \Gamma)_H(x)}) \in [\![(\!|\Gamma|\!)(x)]\!]_{\mathcal{E}}) \wedge (\sum_{x \in dom((\!|\Gamma|\!))} f(x) \leqslant \Phi_{V,H}(\Gamma))$

We choose $f(x)$ as $\Phi_H(V(x) : \Gamma(x))$ for every $x \in dom(\Gamma)$ and it suffices to prove that

- $(\forall x \in dom((\!|\Gamma|\!)). (\Phi_H(V(x) : \Gamma(x)), T, \overline{(V : \Gamma)_H}(x)) \in [\![(\!|\Gamma|\!)(x)]\!]_{\mathcal{E}})$:

  This means given some $x \in dom((\!|\Gamma|\!))$ it suffices to prove that

  $(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V : \Gamma_H)}(x)) \in [\![(\!|\Gamma(x)|\!)]\!]_{\mathcal{E}}$

  From Definition 95 it suffices to prove that

  $(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H,\Gamma(x)}}) \in [\![(\!|\Gamma(x)|\!)]\!]_{\mathcal{E}}$

  From Lemma 97 we know that

  $(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H,\Gamma(x)}}) \in [\![(\!|\Gamma(x)|\!)]\!]$

  And finally from Definition 66 we have

  $(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H,\Gamma(x)}}) \in [\![(\!|\Gamma(x)|\!)]\!]_{\mathcal{E}}$

- $(\sum_{x \in dom((\!|\Gamma|\!))} f(x) \leqslant \Phi_{V,H}(\Gamma))$:

  Since we know that $\Phi_{V,H}(\Gamma) = \sum_{x \in dom(\Gamma)} \Phi_H(V(x) : \Gamma(x))$ therefore we are done

$\square$

Lemma 99 (RAML's stack and its translation are in the cross-lang relation)**.** $\forall H, V, \Gamma.$
$H \models V : \Gamma \implies \forall\, T\, .(T, V, \overline{(V : \Gamma)_H}) \in \lfloor \Gamma \rceil_{\mathcal{V}}^{H}$

Proof. Given some T, it suffices to prove that $(T, V, \overline{(V : \Gamma)_H}) \in \lfloor\Gamma\rceil_{\mathcal{V}}^{H}$

From Definition 85 it suffices to prove that
$\forall x : \tau \in dom(\Gamma).(T, V(x), \overline{(V : \Gamma)_H}(x)) \in \lfloor\tau\rceil_{\mathcal{V}}^{H}$

This means given some $x : \tau \in dom(\Gamma)$ and we need to prove that
$(T, V(x), \overline{(V : \Gamma)_H}(x)) \in \lfloor\tau\rceil_{\mathcal{V}}^{H}$

Since we are given that $H \models V : \Gamma$, it means we have $\forall x \in dom(\Gamma).H \models V(x) \in [\![\Gamma(x)]\!]$

Therefore we get the desired from Lemma 100

□

Lemma 100 (RAML's value and its translation are in the cross-lang relation)**.** $\forall H, {}^s v, \tau.$
$H \models {}^s v \in [\![\tau]\!] \implies \forall\, T\, .(T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in \lfloor\tau\rceil_{\mathcal{V}}^{H}$

Proof. By induction on τ

1. $\tau = unit$:
   To prove: $\forall\, T\, .(T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in \lfloor unit\rceil_{\mathcal{V}}^{H}$
   This means given some T, from Definition 96 it suffices to prove that
   $(T, {}^s v, {}^s v) \in \lfloor unit\rceil_{\mathcal{V}}^{H}$

   We get this directly from Definition 84

2. $\tau = \mathsf{b}$:
   To prove: $\forall\, T\, .(T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in \lfloor \mathsf{b}\rceil_{\mathcal{V}}^{H}$
   This means given some T, from Definition 96 it suffices to prove that
   $(T, {}^s v, !{}^s v) \in \lfloor \mathsf{b}\rceil_{\mathcal{V}}^{H}$

   We get this directly from Definition 84

3. $\tau = L^{\vec{q}}\tau'$:
   By induction on ${}^s v$

   - ${}^s v = NULL$:
     To prove: $\forall\, T\, .(T, NULL, \overline{({}^s v)_{H,\tau}}) \in \lfloor \mathsf{b}\rceil_{\mathcal{V}}^{H}$
     Given some T, from Definition 96 it suffices to prove that
     $(T, NULL, \langle\!\langle (), nil\rangle\!\rangle) \in \lfloor L^{\vec{q}}\tau'\rceil_{\mathcal{V}}^{H}$
     We get this directly from Definition 84
   - ${}^s v = \ell = [{}^s v_1 \ldots {}^s v_n]$:
     To prove: $\forall\, T\, .(T, \ell, \overline{({}^s v)_{H,\tau}}) \in \lfloor \mathsf{b}\rceil_{\mathcal{V}}^{H}$
     Given some T, from Definition 96 it suffices to prove that
     $(T, \ell, \langle\!\langle (), \overline{(H(\ell)\downarrow_1)_{H,\tau'}} :: \overline{(H(\ell)\downarrow_2)_{H,L\tau'}}\rangle\!\rangle) \in \lfloor L^{\vec{q}}\tau'\rceil_{\mathcal{V}}^{H}$
     From Definition 84 it further suffices to prove that
     $(T, H(\ell)\downarrow_1, \overline{(H(\ell)\downarrow_1)_{H,\tau'}}) \in \lfloor\tau'\rceil_{\mathcal{V}} \wedge (T, H(\ell)\downarrow_2, \overline{(H(\ell)\downarrow_2)_{H,L\tau'}}) \in \lfloor L\ \tau'\rceil_{\mathcal{V}}$

     We get $(T, H(\ell)\downarrow_1, \overline{(H(\ell)\downarrow_1)_{H,\tau'}}) \in \lfloor\tau'\rceil_{\mathcal{V}}$ from IH of outer induction
     and $(T, H(\ell)\downarrow_2, \overline{(H(\ell)\downarrow_2)_{H,L\tau'}}) \in \lfloor L\ \tau'\rceil_{\mathcal{V}}$ from IH of inner induction

4. $\tau = (\tau_1, \tau_2)$:

   To prove: $\forall\, T\ .(T, \ell, \overline{(\ell)_{H,(\tau_1,\tau_2)}}) \in \lfloor(\tau_1, \tau_2)\rfloor_{\mathcal{V}}^{H}$

   Given some T, from Definition 96 it suffices to prove that

   $(T, \ell, \langle\!\langle\overline{(H(\ell)\downarrow_1)_{H,\tau_1}}, \overline{(H(\ell)\downarrow_2)_{H,\tau_2}}\rangle\!\rangle) \in \lfloor(\tau_1, \tau_2)\rfloor_{\mathcal{V}}^{H}$

   From Definition 84 it suffices to prove that

   $(T, H(\ell)\downarrow_1, \overline{(H(\ell)\downarrow_1)_{H,\tau_1}}) \in \lfloor\tau_1\rfloor_{\mathcal{V}} \wedge (T, H(\ell)\downarrow_2, \overline{(H(\ell)\downarrow_2)_{H,\tau_2}}) \in \lfloor\tau_2\rfloor_{\mathcal{V}}$

   We get this directly from IH

□

Lemma 101. $\forall {}^s v, {}^t v, \tau, H, T.$
$(T, {}^s v, {}^t v) \in \lfloor\tau\rfloor_{\mathcal{V}}^{H} \implies {}^t v = \overline{({}^s v)_{H,\tau}}$

Proof. Proof by induction on the $\lfloor.\rfloor_{\mathcal{V}}$ relation

1. $\lfloor unit\rfloor_{\mathcal{V}}^{H}$:

   Given: $(T, {}^s v, {}^s v) \in \lfloor unit\rfloor_{\mathcal{V}}^{H}$

   To prove: ${}^s v = \overline{({}^s v)_{H,unit}}$

   Directly from Definition 96

2. $\lfloor \mathsf{b}\rfloor_{\mathcal{V}}^{H}$:

   Given: $(T, {}^s v, !^s v) \in \lfloor \mathsf{b}\rfloor_{\mathcal{V}}^{H}$

   To prove: $!^s v = \overline{({}^s v)_{H,\tau}}$

   Directly from Definition 96

3. $\lfloor(\tau_1, \tau_2)\rfloor_{\mathcal{V}}^{H}$:

   Given: $(T, \ell, \langle\!\langle {}^t v_1, {}^t v_2\rangle\!\rangle) \in \lfloor(\tau_1, \tau_2)\rfloor_{\mathcal{V}}^{H}$

   This means from Definition 84 we have

   $H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in \lfloor\tau_1\rfloor_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in \lfloor\tau_2\rfloor_{\mathcal{V}}$ (R0)

   To prove: $\langle\!\langle {}^t v_1, {}^t v_2\rangle\!\rangle = \overline{(\ell)_{H,(\tau_1,\tau_2)}}$

   From Definition 96 we know that

   $\overline{(\ell)_{H,(\tau_1,\tau_2)}} = \langle\!\langle\overline{(H(\ell)\downarrow_1)_{H,\tau_1}}, \overline{(H(\ell)\downarrow_2)_{H,\tau_2}}\rangle\!\rangle$

   From (R0) we know that $H(\ell)\downarrow_1 = {}^s v_1$ and $H(\ell)\downarrow_2 = {}^s v_2$ therefore we have

   $\overline{(\ell)_{H,(\tau_1,\tau_2)}} = \langle\!\langle\overline{(H(\ell)\downarrow_1)_{H,\tau_1}}, \overline{(H(\ell)\downarrow_2)_{H,\tau_2}}\rangle\!\rangle = \langle\!\langle\overline{{}^s v_1}, \overline{{}^s v_2}\rangle\!\rangle$ (R1)

   Since from (R0) we know that $(T, {}^s v_1, {}^t v_1) \in \lfloor\tau_1\rfloor_{\mathcal{V}}$ therefore we have

   ${}^t v_1 = \overline{{}^s v_1}$ (IH1)

   Similarly we also have

   ${}^t v_2 = \overline{{}^s v_2}$ (IH2)

   We get the desired from IH1, IH2 and (R1)

4. $\lfloor L^{\vec{q}}\tau' \rfloor_{\mathcal{V}}^{H}$:

Given: $(T, \ell_s, \langle\!\langle (), l_t \rangle\!\rangle) \in \lfloor L^{\vec{q}}\tau' \rfloor_{\mathcal{V}}^{H}$ where $(T, \ell_s, l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^{H}$

To prove: $\langle\!\langle (), l_t \rangle\!\rangle = \overline{(\ell_s)_{H,\tau}}$

From Definition 96 we know that
$\overline{(\ell_s)_{H,L^{-}\tau'}} = \langle\!\langle (), \overline{(\ell_s)_{H,L\tau'}} \rangle\!\rangle$

Therefore it suffices to prove that $l_t = \overline{(\ell_s)_{H,L\tau'}}$
We induct on $(T, \ell_s, l_t) \in \lfloor L\tau' \rfloor_{\mathcal{V}}^{H}$

(a) $\ell_s = NULL$:
In this case we know that $l_t = nil$
From Definition 96 we get the desired

(b) $\ell_s = \ell \neq NULL$:
In this case we know that $l_t = {}^{t}v_h :: l'_t$ s.t
$H(\ell) = ({}^{s}v', \ell'_s) \wedge (T, {}^{s}v', {}^{t}v_h) \in \lfloor \tau' \rfloor_{\mathcal{V}} \wedge (T, \ell'_s, l'_t) \in \lfloor L\ \tau' \rfloor_{\mathcal{V}}$
We get the desired from Definition 96, IH of outer induction and IH of inner induction

□

Definition 102 (Top level RAML program translation)**.** Given a top-level RAML program
$P \triangleq F, e_{main}$ where $F \triangleq f1(x) = e_{f1}, \ldots, fn(x) = e_{fn}$ s.t
$\Sigma, x : \tau_{f1} \vdash^{q_1}_{q'_1} e_{f1} : \tau'_{f1}$
$\ldots$
$\Sigma, x : \tau_{fn} \vdash^{q_n}_{q'_n} e_{fn} : \tau'_{fn}$
$\Sigma, \Gamma \vdash^{q}_{q'} e_{main} : \tau$
where $\Sigma = f1 : \tau_{f1} \overset{q_1/q'_1}{\rightarrow} \tau'_{f1}, \ldots, fn : \tau_{fn} \overset{q_n/q'_n}{\rightarrow} \tau'_{fn}$

Translation of P denoted by $\overline{P}$ is defined as $\overline{F}, e_t$ where
$\overline{F} = \mathsf{fix} f_1.\lambda u.\lambda x.e_{t1}, \ldots, \mathsf{fix} f_n.\lambda u.\lambda x.e_{tn}$ s.t
$\Sigma, x : \tau_{f1} \vdash^{q_1}_{q'_1} e_{f1} : \tau'_{f1} \rightsquigarrow e_{t1}$
$\ldots$
$\Sigma, x : \tau_{fn} \vdash^{q_n}_{q'_n} e_{fn} : \tau'_{fn} \rightsquigarrow e_{tn}$
and
$\Sigma, \Gamma \vdash^{q}_{q'} e_{main} : \tau \rightsquigarrow e_t$

Theorem 103 (RAML univariate soundness)**.** $\forall H, H', V, \Gamma, \Sigma, e, \tau, {}^{s}v, p, p', q, q', t.$
$P = F, e$ and $\overline{P}$ be a RAML top-level program and its translation respectively (as defined in Definition 102)
$H \models V : \Gamma \wedge \Sigma, \Gamma \vdash^{q}_{q'} e : \tau \wedge V, H \vdash^{p}_{p'} e \Downarrow_t {}^{s}v, H'$
$\implies$
$p - p' \leqslant (\Phi_{H,V}(\Gamma) + q) - (q' + \Phi_{H}({}^{s}v : \tau))$

Proof. From Definition 102 we are given that
$F \triangleq f_1(x) = e_{f_1}, \ldots, f_n(x) = e_{f_n}$ s.t
$\Sigma, x : \tau_{f_1} \vdash^{q_1}_{q'_1} e_{f_1} : \tau'_{f_1} \rightsquigarrow e_{t_1}$
$\ldots$
$\Sigma, x : \tau_{f_n} \vdash^{q_n}_{q'_n} e_{f_n} : \tau'_{f_n} \rightsquigarrow e_{t_n}$

Let $\forall i \in [1 \dots n].\delta_{sf}(f_i) = (f_i(x) = e_{f_i})$ and $\forall i \in [1 \dots n].\delta_{tf}(f_i) = (\mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i})$

Claim: $\forall\ T\ .(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil_{\mathcal{V}'}^{H}$
Proof.
This means given some T, it suffices to prove that
$(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil_{\mathcal{V}'}^{H}$

We induct on T
Base case: Trivial
Inductive case:
IH: $\forall\ T''<T\ .(T'', \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil_{\mathcal{V}'}^{H}$

From Definition 86 it suffices to prove that
$\forall f_i \in dom(\Sigma).(T, f_i(x) = e_{f_i}\ \delta_{sf}, \mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i}\ \delta_{tf}) \in \lfloor \tau_{f_i} \overset{q_i/q_i'}{\to} \tau_{f_i}' \rceil_{\mathcal{V}'}^{H}$

Given some $f_i \in dom(\Sigma)$ it suffices to prove that
$(T, f_i(x) = e_{f_i}\ \delta_{sf}, \mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i}\ \delta_{tf}) \in \lfloor \tau_{f_i} \overset{q_i/q_i'}{\to} \tau_{f_i}' \rceil_{\mathcal{V}'}^{H}$

From Definition 84 it suffices to prove that
$\forall^s v', {}^t v', T'<T\ .(T', {}^s v', {}^t v') \in \lfloor \tau_{f_i} \rceil_{\mathcal{V}}^{H} \implies (T', e_{f_i}\ \delta_{sf}, e_{t_i}\ \delta_{tf}[()/u][{}^t v'/x][\mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i}\ \delta_{tf}/f_i]) \in \lfloor \tau_{f_1}' \rceil_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$

This means given some ${}^s v', {}^t v', T'<T$ s.t $(T', {}^s v', {}^t v') \in \lfloor \tau_{f_i} \rceil_{\mathcal{V}}^{H}$ it suffices to prove that
$(T', e_{f_i}\ \delta_{sf}, e_{t_i}\ \delta_{tf}[()/u][{}^t v'/x][\mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i}\ \delta_{tf}/f_i]) \in \lfloor \tau_{f_i}' \rceil_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$

Since $\delta_{tf} = \delta_{tf} \cup \{f_i \mapsto \mathsf{fix} f_i.\lambda u.\lambda x.e_{t_i}\ \delta_{tf}\}$, therefore it suffices to prove that
$(T', e_{f_i}\ \delta_{sf}, e_{t_i}\ \delta_{tf}[()/u][{}^t v'/x]) \in \lfloor \tau_{f_i}' \rceil_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$ (C0)
Also since are given $(T', {}^s v', {}^t v') \in \lfloor \tau_{f_i} \rceil_{\mathcal{V}}^{H}$ therefore we have
$(T', \{x \mapsto {}^s v'\}, \{x \mapsto {}^t v'\}) \in \lfloor x : \tau_{f_i} \rceil_{\mathcal{V}}^{H}$

Also from IH we have $(T', \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil_{\mathcal{V}'}^{V,H}$
We can apply Theorem 92 to get
$(T', e_{f_i}\delta_{sf}, e_{t_i}\ ()\ \{x \mapsto {}^t v'\}\delta_{tf}) \in \lfloor \tau_{f_i}' \rceil_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$
And this prove (C0)

$\square$

From Theorem 82 we know that $\exists e_t$ s.t
$\Sigma, \Gamma \vdash_{q'}^{q} e : \tau \rightsquigarrow e_t$ and $.; .; (\!|\Sigma|\!); (\!|\Gamma|\!) \vdash e_t : [q]\, \mathbf{1} \multimap \mathbb{M}\, 0\, [q']\, (\!|\tau|\!)$

From Lemma 99 we know that $\forall\ T\ .(T, V, \overline{(V : \Gamma)_H}) \in \lfloor \Gamma \rceil_{\mathcal{V}}^{H}$

Also from the Claim proved above we know that $\forall\ T\ .(T, \delta_{sf}, \delta_{tf}) \in \lfloor \Sigma \rceil_{\mathcal{V}'}^{H}$
Therefore from Theorem 92 we know that $\forall\ T\ .(T, e\delta_{sf}, e_t\ ()\ \overline{(V : \Gamma)_H}\delta_{tf}) \in \lfloor \tau \rceil_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have
$\forall\ T\ .\forall H_1', {}^s v_1, p_1, p_1', t' <T\ .\ V, H \vdash_{p_1'}^{p_1} e\delta_{sf} \Downarrow_{t'} {}^s v_1, H_1' \implies \exists^t v_t, {}^t v_f, J.e_t\ ()\ \overline{(V : \Gamma)_H}\delta_{tf} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t', {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H_1'} \wedge p_1 - p_1' \leqslant J$ (RD-0.0)

We are given that $V, H \vdash_{p'}^{p} e \Downarrow_t {}^s v, H'$
Therefore instantiating (RD-0.0) with $t+1, H', {}^s v, p, p', t$ we get
$\exists^t v_t, {}^t v_f, J.e_t\ ()\ \overline{(V : \Gamma)_H}\delta_{tf} \Downarrow_{-} {}^t v_t \Downarrow_{-}^{J} {}^t v_f \wedge (1, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'} \wedge p - p' \leqslant J$ (RD-0)

From reduction rules we know that $\exists t_1, t_2$ s.t $e_t\ ()\ \overline{(V : \Gamma)_H}\delta_{tf} \Downarrow_{t_1} {}^t v_t \Downarrow_{t_2}^{J} {}^t v_f$

Since from Lemma 98 we know that $\forall\, T\,.(\Phi_{V,H}(\Gamma), T, \overline{(V:\Gamma)_H}) \in [\![(\!|\Gamma|\!)]\!]$
Therefore we also have $(\Phi_{V,H}(\Gamma), t_1+t_2+1, \overline{(V:\Gamma)_H}) \in [\![(\!|\Gamma|\!)]\!]$

Therefore from Theorem 80 we get
$\exists p_v.(p_v, 1, {}^t v_f) \in [\![(\!|\tau|\!)]\!] \;\wedge\; J \leqslant (q+\Phi_{V,H}(\Gamma)) - (q'+p_v)$ (RD-1)

Since we have $(1, {}^s v, {}^t v_f) \in \lfloor \tau \rceil_{\mathcal{V}}^{H'}$ therefore from Lemma 101 we know that ${}^t v_f = \overline{({}^s v)_{H',\tau}}$

From Lemma 97 we know that $\forall\, T\,.(\Phi_H({}^s v:\tau), T, \overline{({}^s v)_{H',\tau}}) \in [\![(\!|\tau|\!)]\!]$

Therefore we have $(\Phi_H({}^s v:\tau), 1, \overline{({}^s v)_{H',\tau}}) \in [\![(\!|\tau|\!)]\!]$ (RD-2)

From (RD-1), (RD-2) and Lemma 69 we know that $p_v \geqslant \Phi_H({}^s v:\tau)$

Since from (RD-1) we know that $J \leqslant (q+\Phi_{V,H}(\Gamma)) - (q'+p_v)$ therefore we also have
$J \leqslant (q+\Phi_{V,H}(\Gamma)) - (q'+\Phi_H({}^s v:\tau))$ (RD-3)

Finally from (RD-0) and (RD-3) we get the desired.

□

3 Examples

3.1 Strict functional queue

$enqueue : \forall m,n.\,[3]\,\mathbf{1} \multimap \tau \multimap L^n([2]\,\tau) \multimap L^m\tau \multimap \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$enqueue \triangleq \Lambda.\lambda\; p\; a\; l_1\; l_2.\mathsf{release} - = p \mathsf{\ in\ bind}\, x = \mathsf{store}\, a \mathsf{\ in\ bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$

Typing derivation for enqueue $enqueue$

$T_0 = \forall m,n.\,[3]\,\mathbf{1} \multimap \tau \multimap L^n([2]\,\tau) \multimap L^m\tau \multimap \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_1 = [3]\,\mathbf{1} \multimap \tau \multimap L^n([2]\,\tau) \multimap L^m\tau \multimap \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_{1.0} = [3]\,\mathbf{1}$
$T_2 = \tau \multimap L^n([2]\,\tau) \multimap L^m\tau \multimap \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_3 = L^n([2]\,\tau) \multimap L^m\tau \multimap \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_{3.1} = L^n([2]\,\tau)$
$T_{3.2} = L^m\tau$
$T_4 = \mathbb{M}\,0\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_5 = \mathbb{M}\,1\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$
$T_6 = \mathbb{M}\,3\,(L^{n+1}([2]\,\tau) \otimes L^m\tau)$

$enqueue = \Lambda.\lambda\; p\; a\; l_1\; l_2.\mathsf{release} - = p \mathsf{\ in\ bind}\, x = \mathsf{store}\, a \mathsf{\ in\ bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$
$E_1 = \lambda\; p\; a\; l_1\; l_2.\mathsf{release} - = p \mathsf{\ in\ bind}\, x = \mathsf{store}\, a \mathsf{\ in\ bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$
$E_2 = \mathsf{release} - = p \mathsf{\ in\ bind}\, x = \mathsf{store}\, a \mathsf{\ in\ bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$
$E_3 = \mathsf{bind}\, x = \mathsf{store}\, a \mathsf{\ in\ bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$
$E_4 = \mathsf{bind} - = \uparrow^1 \mathsf{in\ ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$
$E_5 = \mathsf{ret}\langle\!\langle (x :: l_1), l_2\rangle\!\rangle$

D2:

$$\frac{}{.; m,n; .; .; x:[2]\,\tau, l_1 : L^n([2]\,\tau), l_2 : L^m\tau \vdash E_5 : T_4}$$

D1:

$$\frac{\dfrac{}{.; m,n; .; .; . \vdash \uparrow^1 : \mathbb{M}\,1\,\mathbf{1}} \qquad D2}{.; m,n; .; .; x:[2]\,\tau, l_1 : L^n([2]\,\tau), l_2 : L^m\tau \vdash E_4 : T_5}$$

D0:

$$\dfrac{\dfrac{}{.;m,n;.;.;a:\tau\vdash \mathsf{store}\,a:\mathbb{M}\,2\,([2]\,\tau)} \quad D1}{.;m,n;.;.;a:\tau,l_1:L^n([2]\,\tau),l_2:L^m\tau\vdash E_3:T_6}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.;m,n;.;.;p:T_{1.0}\vdash p:T_{1.0}} \quad D0}{.;m,n;.;.;p:T_{1.0},a:\tau,l_1:L^n([2]\,\tau),l_2:L^m\tau\vdash E_2:T_4}}{.;m,n;.;.;.\vdash E_1:T_1}}{.;.;.;.;.\vdash \mathit{enqueue}:T_0}$$

$Dq : \forall m,n.(m+n>0)\Rightarrow [1]\,\mathbf{1}\multimap L^m([2]\,\tau)\multimap L^n\tau\multimap$
$\mathbb{M}\,0\,(\exists m',n'.((m'+n'+1)=(m+n))\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$

$Dq \triangleq \Lambda.\Lambda.\Lambda.\lambda\ p\ l_1\ l_2.\mathsf{match}\ l_2\ \mathsf{with}\ |nil\mapsto E_1\ |h_2::l_2'\mapsto E_2$
$E_1 = \mathsf{bind}\,l_r = M\ [][]\ l_1\ nil\ \mathsf{in\ match}\ l_r\ \mathsf{with}\ |nil\mapsto -\ |h_r::l_r'\mapsto E_{1.1}$
$E_{1.1} = \mathsf{release} - = p\ \mathsf{in\ bind} - = \uparrow^1\ \mathsf{in\ ret}\,\Lambda.\langle\langle nil,l_r'\rangle\rangle$
$E_2 = \mathsf{release} - = p\ \mathsf{in\ bind} - = \uparrow^1\ \mathsf{in\ ret}\,\Lambda.\langle\langle l_1,l_2'\rangle\rangle$

Typing derivation for dequeue Dq

$T_0 = \forall m,n.(m+n>0)\Rightarrow [1]\,\mathbf{1}\multimap L^m([2]\,\tau)\multimap L^n\tau\multimap$
$\mathbb{M}\,0\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_1 = (m+n>0)\Rightarrow [1]\,\mathbf{1}\multimap L^m([2]\,\tau)\multimap L^n\tau\multimap$
$\mathbb{M}\,0\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_2 = [1]\,\mathbf{1}\multimap L^m([2]\,\tau)\multimap L^n\tau\multimap \mathbb{M}\,0\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_{2.1} = L^m([2]\,\tau)$
$T_3 = L^n\tau\multimap \mathbb{M}\,0\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_{3.1} = L^n\tau$
$T_4 = \mathbb{M}\,0\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_{4.1} = \mathbb{M}\,1\,(\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_5 = (\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))$
$T_{5.1} = (\exists m',n'.(m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau))[m/m'][i/n']$
$T_{5.2} = (L^m[2]\,\tau\otimes L^n\tau)$
$T_6 = (m'+n'+1)=(m+n)\&(L^{m'}[2]\,\tau\otimes L^{n'}\tau)[0/m'][i/n']$
$T_7 = (L^0[2]\,\tau\otimes L^{n-1}\tau)$

$E_0 = \Lambda.\Lambda.\Lambda.\lambda\ l_1\ l_2.\mathsf{match}\ l_2\ \mathsf{with}\ |nil\mapsto E_1\ |h_2::l_2'\mapsto E_2$
$E_{0.1} = \lambda\ p\ l_1\ l_2.\mathsf{match}\ l_2\ \mathsf{with}\ |nil\mapsto E_1\ |h_2::l_2'\mapsto E_2$
$E_{0.2} = \mathsf{match}\ l_2\ \mathsf{with}\ |nil\mapsto E_1\ |h_2::l_2'\mapsto E_2$
$E_1 = \mathsf{bind}\,l_r = M\ [][]\ l_1\ nil\ \mathsf{in\ match}\ l_r\ \mathsf{with}\ |nil\mapsto -\ |h_r::l_r'\mapsto E_{1.1}$
$E_{1.1} = \mathsf{release} - = p\ \mathsf{in\ bind}\,x = \uparrow^1\ \mathsf{in}\ \Lambda.\,\mathsf{ret}\langle\langle nil,l_r'\rangle\rangle$
$E_2 = \mathsf{release} - = p\ \mathsf{in\ bind}\,x = \uparrow^1\ \mathsf{in}\ \Lambda.\,\mathsf{ret}\langle\langle l_1,l_2'\rangle\rangle$

D1.3:

$$\dfrac{}{.;m,n;(n>0),(m+n)>0;.;h_2:\tau,l_2':L^{m-1}\tau,l_1:T_{2.1}\vdash\langle\langle l_1,l_2'\rangle\rangle:T_{5.2}}$$

D1.2:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.; m, n; (n > 0), (m + n) > 0 \models (m + i + 1) = (m + n)} \qquad D1.3}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{m-1}\tau, l_1 : T_{2.1} \vdash \Lambda.\langle\!\langle l_1, l_2' \rangle\!\rangle : T_{5.1}}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{m-1}\tau, l_1 : T_{2.1} \vdash \Lambda.\langle\!\langle l_1, l_2' \rangle\!\rangle : T_5}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{m-1}\tau, l_1 : T_{2.1} \vdash \mathsf{ret}\, \Lambda.\langle\!\langle l_1, l_2' \rangle\!\rangle : T_4}$$

D1.1:

$$\dfrac{\dfrac{}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{n-1}\tau \vdash \uparrow^1 : \mathbb{M}\, 1\, \mathbf{1}} \qquad D1.2}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{n-1}\tau, l_1 : T_{2.1} \vdash \mathsf{bind} - = \uparrow^1 \mathsf{~in~ret}\, \Lambda.\langle\!\langle l_1, l_2' \rangle\!\rangle : T_{4.1}}$$

D1:

$$\dfrac{\dfrac{\dfrac{}{.; m, n; (n > 0), (m + n) > 0; .; p : [1]\, \mathbf{1} \vdash p : [1]\, \mathbf{1}} \qquad D1.1}{\begin{array}{c}.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{n-1}\tau, l_1 : T_{2.1} \vdash \\ \mathsf{release} - = p \mathsf{~in~bind} - = \uparrow^1 \mathsf{~in~ret}\, \Lambda.\langle\!\langle l_1, l_2' \rangle\!\rangle : T_4\end{array}}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l_2' : L^{n-1}\tau, l_1 : T_{2.1} \vdash E_2 : T_4}$$

D0.05:

$$\dfrac{}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \langle\!\langle nil, l_r' \rangle\!\rangle : T_7}$$

D0.04:

$$\dfrac{\dfrac{\dfrac{D0.05}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \Lambda.\langle\!\langle nil, l_r' \rangle\!\rangle : T_6}}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \Lambda.\langle\!\langle nil, l_r' \rangle\!\rangle : T_5}}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \mathsf{ret}\, \Lambda.\langle\!\langle nil, l_r' \rangle\!\rangle : T_4}$$

D0.03:

$$\dfrac{\dfrac{}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \uparrow^1 : \mathbb{M}\, 1\, \mathbf{1}} \qquad D0.04}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau \vdash \mathsf{bind} - = \uparrow^1 \mathsf{~in~ret}\, \Lambda.\langle\!\langle nil, l_r' \rangle\!\rangle : T_{4.1}}$$

D0.02:

$$\dfrac{\dfrac{}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; p : [1]\, \mathbf{1} \vdash p : [1]\, \mathbf{1}} \qquad D0.03}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l_r' : L^{m-1}\tau, p : [1]\, \mathbf{1} \vdash E_{1.1} : T_4}$$

D0.01:

$$\dfrac{}{.; m, n; (n = 0), (m + n) > 0; .; . \vdash \mathsf{fix}\ x.x : T_4}$$

D0.0:

$$\dfrac{\dfrac{}{.; m, n; (n = 0), (m + n) > 0; .; l_r : L^m\tau \vdash l_r : L^m\tau} \qquad D0.01 \qquad D0.02}{.; m, n; (n = 0), (m + n) > 0; .; l_r : L^m\tau, p : [1]\, \mathbf{1} \vdash \mathsf{match}\ l_r \mathsf{~with~} |nil \mapsto - \ |h_r :: l_r' \mapsto E_{1.1} : T_4}$$

D0:

$$\dfrac{\dfrac{}{.; m, n; (n = 0), (m + n) > 0; .; l_1 : T_{2.1} \vdash M\ [][]\ l_1\ nil : \mathbb{M}\,0\,(L^m\tau)} \quad D0.0}{.; m, n; (n = 0), (m + n) > 0; .; l_1 : T_{2.1}, p : [1]\,\mathbf{1} \vdash E_1 : T_4}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{.; m, n; (m + n) > 0; .; l_2 : T_{3.1} \vdash l_2 : T_{3.1}} \quad D0 \quad D1}{.; m, n; (m + n) > 0; .; l_1 : T_{2.1}, l_2 : T_{3.1}, p : [1]\,\mathbf{1} \vdash E_{0.2} : T_0}}{.; m, n; (m + n) > 0; .; . \vdash E_{0.1} : T_0}}{.; .; .; .; . \vdash E_0 : T_0}$$

$Move : \forall m, n.L^m([2]\,\tau) \multimap L^n\tau \multimap \mathbb{M}\,0\,(L^{m+n}\tau)$
$Move \triangleq \mathsf{fix} M \Lambda.\Lambda.\lambda\ l_1\ l_2.\mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_1\ |h_1 :: l_1' \mapsto E_2$
$E_1 = \mathsf{ret}(l_2)$
$E_2 = \mathsf{release}\, h_1' = h_1\ \mathsf{in}\ \mathsf{bind} - = \uparrow^2\ \mathsf{in}\ M\ [][]\ l_1'\ (h_1 :: l_2)$

Typing derivation for $Move$

$T_0 = \forall m, n.L^m([2]\,\tau) \multimap L^n\tau \multimap \mathbb{M}\,0\,(L^{m+n}\tau)$
$T_1 = L^m([2]\,\tau) \multimap L^n\tau \multimap \mathbb{M}\,0\,(L^{m+n}\tau)$
$T_{1.1} = L^m([2]\,\tau)$
$T_2 = L^n\tau \multimap \mathbb{M}\,0\,(L^{m+n}\tau)$
$T_{2.1} = L^n\tau$
$T_3 = \mathbb{M}\,0\,(L^{m+n}\tau)$
$T_5 = \mathbb{M}\,2\,(L^{m+n}\tau)$

$E_0 = \mathsf{fix}\ M.\Lambda.\Lambda.\lambda\ l_1\ l_2.\mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_1\ |h_1 :: l_1' \mapsto E_2$
$E_{0.0} = \Lambda.\Lambda.\lambda\ l_1\ l_2.\mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_1\ |h_1 :: l_1' \mapsto E_2$
$E_{0.1} = \lambda\ l_1\ l_2.\mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_1\ |h_1 :: l_1' \mapsto E_2$
$E_{0.2} = \mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_1\ |h_1 :: l_1' \mapsto E_2$
$E_1 = \mathsf{ret}(l_2)$
$E_2 = \mathsf{release} - = h\ \mathsf{in}\ \mathsf{bind} - = \uparrow^2\ \mathsf{in}\ M\ [][]\ l_1'\ (h_1 :: l_2)$
$E_{2.1} = \mathsf{bind} - = \uparrow^2\ \mathsf{in}\ M\ [][]\ l_1'\ (h_1 :: l_2)$
$E_{2.2} = M\ [][]\ l_1'\ (h_1 :: l_2)$

D3:

$$\dfrac{}{.; m, n; (m > 0); M : T_0; l_1' : L^{m-1}[2]\,\tau, l_2 : T_{2.1}, h_1' : \tau \vdash M\ [][]\ l_1'\ (h_1' :: l_2) : T_3}$$

D2:

$$\dfrac{\dfrac{}{.; m, n; (m > 0); M : T_0; . \vdash \uparrow^2 : \mathbb{M}\,2\,\mathbf{1}} \quad D3}{.; m, n; (m > 0); M : T_0; l_1' : L^{m-1}[2]\,\tau, l_2 : T_{2.1}, h_1' : \tau \vdash E_{2.1} : T_5}$$

D1:

$$\dfrac{\dfrac{}{.; m, n; (m > 0); M : T_0; h_1 : [2]\,\tau \vdash h_1 : [2]\,\tau} \quad D2}{.; m, n; (m > 0); M : T_0; h_1 : [2]\,\tau, l_1' : L^{m-1}[2]\,\tau, l_2 : T_{2.1} \vdash E_2 : T_3}$$

D0:

$$\frac{}{.;m,n;m=0;M:T_0;l_2:T_{2.1}\vdash E_1:T_3}$$

Main derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{.;m,n;.;M:T_0;l_1:T_{1.1}\vdash l_1:T_{1.1}}\quad D0\quad D1}{.;m,n;.;M:T_0;l_1:T_{1.1},l_2:T_{2.1}\vdash E_{0.2}:T_1}}{.;m,n;.;M:T_0;.\vdash E_{0.2}:T_1}}{.;.;.;M:T_0;.\vdash E_{0.1}:T_{0.0}}}{.;.;.;.;.\vdash E_0:T_0}}{.;.;.;.;.\vdash Move:T_0}$$

3.2 Church numerals

$\mathsf{Nat} = \lambda_t n.\forall\alpha:\mathbb{N}\to Type.\forall C:\mathbb{N}\to\mathbb{N}.$
$!(\forall j_n.((\alpha\ j_n\otimes[C\ j_n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ (j_n+1))))\multimap\mathbb{M}\,0\,((\alpha\ 0\otimes[(\sum_{i<n}C\ i)+n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ n))$

$e_1\ \uparrow^1 e_2 \triangleq \mathsf{bind}\ -=\uparrow^1\ \mathsf{in}\ e_1\ e_2$

$$\frac{\frac{}{\Psi;\Theta;\Delta;\Omega_1;\Gamma_1\vdash e_1:\tau_1\multimap\mathbb{M}(n)\,\tau_2}\qquad\frac{}{\Psi;\Theta;\Delta;\Omega_2;\Gamma_2\vdash e_2:\tau_1}}{\Psi;\Theta;\Delta;\Omega_1\oplus\Omega_2;\Gamma_1\oplus\Gamma_2\vdash e_1\ \uparrow^1 e_2:\mathbb{M}(n+1)\,\tau_2}$$

Type derivation for $\overline{0}$

$\overline{0} = \Lambda.\Lambda.\lambda f.\,\mathsf{ret}\,\lambda x.\,\mathsf{let}\langle\langle y_1,y_2\rangle\rangle = x\ \mathsf{in}\ \mathsf{ret}\,y_1 : \mathsf{Nat}\ 0$

$T_0 =$
$\forall\alpha.\forall C.!(\forall j_n.((\alpha\ j_n\otimes[C\ j_n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ (j_n+1))))\multimap\mathbb{M}\,0\,((\alpha\ 0\otimes[0]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ 0))$
$T_{0.1} = \forall C.!(\forall j_n.((\alpha\ j_n\otimes[C\ j_n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ (j_n+1))))\multimap\mathbb{M}\,0\,((\alpha\ 0\otimes[0]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ 0))$
$T_{0.2} = !(\forall j_n.((\alpha\ j_n\otimes[C\ j_n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ (j_n+1))))\multimap\mathbb{M}\,0\,((\alpha\ 0\otimes[0]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ 0))$
$T_{0.3} = !(\forall j_n.((\alpha\ j_n\otimes[C\ j_n]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ (j_n+1))))$
$T_1 = \mathbb{M}\,0\,((\alpha\ 0\otimes[0]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ 0))$
$T_{1.1} = ((\alpha\ 0\otimes[0]\,\mathbf{1})\multimap\mathbb{M}\,0\,(\alpha\ 0))$
$T_2 = (\alpha\ 0\otimes[0]\,\mathbf{1})$
$T_{2.1} = \alpha\ 0$
$T_{2.2} = [0]\,\mathbf{1}$
$T_3 = \mathbb{M}\,0\,(\alpha\ 0)$
$TI = \alpha:\mathbb{N}\to Type; C:\mathbb{N}\to Sort$
D1:

$$\frac{}{TI;.;.;f:T_{0.3},y_1:T_{2.1},y_2:T_{2.2}\vdash \mathsf{ret}\,y_1:\mathbb{M}\,0\,T_{2.1}}$$

D0:

$$\frac{}{TI;.;.;f:T_{0.3},x:T_2\vdash x:T_2}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D0 \qquad D1}{TI; .; .; f : T_{0.3}, x : T_2 \vdash \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1 : T_3}}{TI; .; .; f : T_{0.3} \vdash \lambda x.\, \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1 : T_{1.1}}}{TI; .; .; f : T_{0.3} \vdash \mathsf{ret}\, \lambda x.\, \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1 : T_1}}{TI; .; .; . \vdash \lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1 : T_{0.2}}}{\alpha : \mathbb{N} \to Type; .; .; ; .; . \vdash \Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1 : T_{0.1}}}{.; .; .; .; . \vdash \Lambda.\Lambda.(\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{ret}\, y_1) : T_0}$$

Type derivation for $\overline{1}$

$\overline{1} = \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\, !\, f_u = f \text{ in } \mathsf{let}\langle\langle y_1, y_2\rangle\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_1 : \mathsf{Nat}\ 1$
where
$E_1 = \mathsf{bind}\, a = \mathsf{store}() \text{ in } f_u\ []\ \uparrow^1\langle\langle y_1, a\rangle\rangle$

$T_0 = \forall \alpha : \mathbb{N} \to Type.\forall C : \mathbb{N} \to Sort.$
$!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ 1))$
$T_{0.1} = \forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ 1))$
$T_{0.2} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ 1))$
$T_{0.3} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1))))$
$T_{0.4} = (\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1))))$
$T_{0.5} = (\alpha\ 0 \otimes [C\ 0]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (0 + 1))$
$T_1 = \mathbb{M}\, 0\, ((\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ 1))$
$T_{1.1} = ((\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ 1))$
$T_2 = (\alpha\ 0 \otimes [C\ 0 + 1]\, \mathbf{1})$
$T_{2.1} = \alpha\ 0$
$T_{2.2} = [C\ 0 + 1]\, \mathbf{1}$
$T_3 = \mathbb{M}\, 0\, (\alpha\ 1)$
$TI = \alpha : \mathbb{N} \to Type; C : \mathbb{N} \to Sort$
D7:

$$\dfrac{}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, a : [C\ 0]\, \mathbf{1} \vdash \langle\langle y_1, a\rangle\rangle : (T_{2.1} \otimes [C\ 0]\, \mathbf{1})}$$

D6:

$$\dfrac{}{TI; .; f_u : T_{0.4}; . \vdash f_u\ [] : T_{0.5}}$$

D5:

$$\dfrac{D6 \qquad D7}{TI; .; f_u : T_{0.4}; y_1 : T_{2.2}, a : [C\ 0]\, \mathbf{1} \vdash f_u\ []\ \uparrow^1\langle\langle y_1, a\rangle\rangle : \mathbb{M}\, 1\, \alpha\ 1}$$

D4:

$$\dfrac{\dfrac{D4}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \mathsf{store}() : \mathbb{M}(C\ 0)\, [C\ 0]\, \mathbf{1}} \qquad D5}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \mathsf{bind}\, a = \mathsf{store}() \text{ in } f_u\ []\ \uparrow^1\langle\langle y_1, a\rangle\rangle : \mathbb{M}(C\ 0 + 1)\, \alpha\ 1}$$

D3:

$$\dfrac{D4}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1} \vdash E_1 : \mathbb{M}(C\ 0 + 1)\, \alpha\ 1}$$

D2:

$$\dfrac{\dfrac{}{TI; .; f_u : T_{0.4}; y_2 : T_{2.2} \vdash y_2 : T_{2.2}} \qquad D3}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_3}$$

D1:

$$\dfrac{\dfrac{}{TI; .; f_u : T_{0.4}; x : T_2 \vdash x : T_2} \qquad D2}{TI; .; f_u : T_{0.4}; x : T_2 \vdash \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_3}$$

D0:

$$\dfrac{}{TI; .; .; f : T_{0.3} \vdash f : T_{0.3}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D0 \qquad D1}{TI; .; .; f : T_{0.3}, x : T_2 \vdash \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_3}}{TI; .; .; f : T_{0.3} \vdash \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_{1.1}}}{TI; .; .; f : T_{0.3} \vdash \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_1}}{TI; .; .; . \vdash \lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_{0.2}}}{.; \alpha : \mathbb{N} \to Type; .; . \vdash \Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_{0.1}}}{.; .; .; . \vdash \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; E_1 : T_0}$$

<u>Type derivation for $\overline{2}$</u>

$\overline{2} = \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \;\mathsf{in}\; \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \;\mathsf{in}\; \mathsf{release} - = y_2 \;\mathsf{in}\; \mathsf{bind}\, b = E_1 \;\mathsf{in}\; E_2 : \mathsf{Nat}\ 2$
where
$E_1 = \mathsf{bind}\, a = \mathsf{store}() \;\mathsf{in}\; f_u\ []\ \uparrow^1\langle\!\langle y_1, a \rangle\!\rangle$
$E_2 = \mathsf{bind}\, c = \mathsf{store}() \;\mathsf{in}\; f_u\ []\ \uparrow^1\langle\!\langle b, c \rangle\!\rangle$

$T_0 =$
$\forall \alpha : \mathbb{N} \to Type.\forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2))$
$T_{0.1} = \forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2))$
$T_{0.2} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1)))) \multimap ((\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2))$
$T_{0.3} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1))))$
$T_{0.4} = (\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1))))$
$T_{0.5} = (\alpha\ 0 \otimes [C\ 0]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 1)$
$T_{0.6} = (\alpha\ 1 \otimes [C\ 1]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2)$
$T_1 = \mathbb{M}\,0\,((\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2))$
$T_{1.1} = ((\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ 2))$
$T_2 = (\alpha\ 0 \otimes [C\ 0 + C\ 1 + 2]\,\mathbf{1})$
$T_{2.1} = \alpha\ 0$
$T_{2.2} = [C\ 0 + C\ 1 + 2]\,\mathbf{1}$
$T_3 = \mathbb{M}\,1\,(\alpha\ 2)$
$T_{3.1} = \mathbb{M}(C\ 0 + C\ 1 + 2)\,(\alpha\ 2)$
$TI = \alpha : \mathbb{N} \to Type; C : \mathbb{N} \to Sort$

D5.22

$$\frac{}{TI; .; f_u : T_{0.4}; b : \alpha\ 1, c : [(C\ 1)]\,\mathbf{1} \vdash \langle\!\langle b, c \rangle\!\rangle : (\alpha\ 1 \otimes [(C\ 1)]\,\mathbf{1})}$$

D5.21

$$\frac{}{TI; .; f_u : T_{0.4}; . \vdash f_u\ [] : T_{0.6}}$$

D5.2

$$\frac{D5.21 \qquad D5.22}{TI; .; f_u : T_{0.4}; b : \alpha\ 1, c : [(C\ 1)]\,\mathbf{1} \vdash f_u\ []\ \uparrow^1 \langle\!\langle b, c \rangle\!\rangle : T_3}$$

D5.1

$$\frac{}{TI; .; f_u : T_{0.4}; . \vdash \mathsf{store}() : \mathbb{M}(C\ 1)\,[(C\ 1)]\,\mathbf{1}}$$

D5:

$$\frac{\dfrac{D5.1 \qquad D5.2}{TI; .; f_u : T_{0.4}; b : \alpha\ 1 \vdash \mathsf{bind}\, c = \mathsf{store}()\ \mathsf{in}\ f_u\ []\ \langle\!\langle b, c \rangle\!\rangle : \mathbb{M}(C\ 1+1)\,(\alpha\ 2)}}{TI; .; f_u : T_{0.4}; b : \alpha\ 1 \vdash E_2 : \mathbb{M}(C\ 1+1)\,(\alpha\ 2)}$$

D4.12:

$$\frac{}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, a : [(C\ 0)]\,\mathbf{1} \vdash \langle\!\langle y_1, a \rangle\!\rangle : (T_{2.1} \otimes [(C\ 0)]\,\mathbf{1})}$$

D4.11:

$$\frac{}{TI; .; f_u : T_{0.4}; . \vdash f_u\ [] : T_{0.5}}$$

D4.1:

$$\frac{D4.11 \qquad D4.12}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, a : [(C\ 0)]\,\mathbf{1} \vdash f_u\ []\ \uparrow^1 \langle\!\langle y_1, a \rangle\!\rangle : \mathbb{M}\,1\,(\alpha\ 1)}$$

D4:

$$\frac{\dfrac{\dfrac{}{TI; .; f_u : T_{0.4}; . \vdash \mathsf{store}() : \mathbb{M}(C\ 0)\,[(C\ 0)]\,\mathbf{1}} \qquad D4.1}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1} \vdash \mathsf{bind}\, a = \mathsf{store}()\ \mathsf{in}\ f_u\ []\ \uparrow^1\langle\!\langle y_1, a \rangle\!\rangle : \mathbb{M}(C\ 0+1)\,(\alpha\ 1)}}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1} \vdash E_1 : \mathbb{M}(C\ 0+1)\,(\alpha\ 1)}$$

D3.2:

$$\frac{D4 \qquad D5}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1} \vdash \mathsf{bind}\, b = E_1\ \mathsf{in}\ E_2 : T_{3.1}}$$

D3.1:

$$\frac{}{TI; .; f_u : T_{0.4}; y_2 : T_{2.2} \vdash y_2 : T_{2.2}}$$

D3:

$$\frac{D3.1 \qquad D3.2}{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \mathsf{release} - = y_2\ \mathsf{in}\ \mathsf{bind}\, b = E_1\ \mathsf{in}\ E_2 : T_3}$$

D2:

$$\frac{}{TI; .; f_u : T_{0.4}; x : T_2 \vdash x : T_2}$$

D1:

$$\frac{D2 \qquad D3}{TI; .; f_u : T_{0.4}; x : T_2 \vdash \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_3}$$

D0:

$$\frac{}{TI; .; .; f : T_{0.3} \vdash f : T_{0.3}}$$

D0.0:

$$\frac{\dfrac{\dfrac{\dfrac{D0 \qquad D1}{\begin{array}{c} TI; .; .; f : T_{0.3}, x : T_2 \vdash \\ \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_3 \end{array}}}{\begin{array}{c} TI; .; .; f : T_{0.3} \vdash \\ \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_{1.1} \end{array}}}{\begin{array}{c} TI; .; .; f : T_{0.3} \vdash \\ \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_1 \end{array}}}{\begin{array}{c} TI; .; .; . \vdash \\ \lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_{0.2} \end{array}}$$

Main derivation:

$$\frac{\dfrac{D0.0}{\begin{array}{c} .; \alpha : \mathbb{N} \to Type; .; . \vdash \\ \Lambda C.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_{0.1} \end{array}}}{.; .; .; . \vdash \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b = E_1 \ \mathsf{in}\ E_2 : T_0}$$

Type derivation for $succ : \forall n.\, [2]\, \mathbf{1} \multimap \mathbb{M}\, 0\, (\mathsf{Nat}\ n \multimap \mathbb{M}\, 0\, (\mathsf{Nat}\ (n+1)))$

$succ = \Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N}.\, \mathsf{ret}\, \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \ \mathsf{in}\ \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \ \mathsf{in}\ \mathsf{release} - = y_2 \ \mathsf{in}\ E_0$
where
$E_0 = \mathsf{release} - = p \ \mathsf{in}\ \mathsf{bind}\, a = E_1 \ \mathsf{in}\ E_2$
$E_1 = \mathsf{bind}\, b = \mathsf{store}() \ \mathsf{in}\ \mathsf{bind}\, b_1 = (\overline{N}\ []\ []\ \uparrow^1 !f_u) \ \mathsf{in}\ b_1\ \uparrow^1 \langle\!\langle y_1, b \rangle\!\rangle$
$E_2 = \mathsf{bind}\, c = \mathsf{store}() \ \mathsf{in}\ \mathsf{ret}\, f_u\ []\ \uparrow^1 \langle\!\langle a, c \rangle\!\rangle$

$T_p = [2]\, \mathbf{1}$
$T_0 = \forall n.T_p \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n] \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n+1]))$
$T_{0.0} = T_p \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n] \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n+1]))$
$T_{0.01} = \mathbb{M}\, 0\, (\mathsf{Nat}[n] \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n+1]))$
$T_{0.1} = \mathsf{Nat}[n] \multimap \mathbb{M}\, 0\, (\mathsf{Nat}[n+1])$
$T_{0.2} = \mathbb{M}\, 0\, (\mathsf{Nat}[n+1])$
$T_{0.11} = \mathsf{Nat}[n]$
$T_{0.12} =$
$\forall \alpha : \mathbb{N} \to Type.\forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap$
$\mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap \mathbb{M}\, 0\, ((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ n))$
$T_{0.13} = \forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap$
$\mathbb{M}\, 0\, ((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ n))$
$T_{0.14} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ (j_n + 1)))) \multimap$
$\mathbb{M}\, 0\, ((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ n))$
$T_{0.15} = \mathbb{M}\, 0\, ((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\, \mathbf{1}) \multimap \mathbb{M}\, 0\, (\alpha\ n))$

$T_{0.151} = \mathbb{M}\,1\,((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ n))$

$T_{0.16} = ((\alpha\ 0 \otimes [C\ 0 + \ldots + C\ (n-1) + n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ n))$

$T_{0.2} = \mathsf{Nat}[n+1]$

$T_1 =$

$\forall \alpha : \mathbb{N} \to Type.\forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1)))) \multimap$
$\mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n+1)))$

$T_{1.1} = \forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1)))) \multimap$
$\mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n+1)))$

$T_{1.2} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1)))) \multimap$
$\mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n+1)))$

$T_{1.3} = !(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1))))$

$T_{1.31} = (\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1))))$

$T_{1.40} = \mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n+1)))$

$T_{1.4} = ((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n+1)))$

$T_{1.41} = (\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1})$

$T_{1.411} = \alpha\ 0$

$T_{1.412} = [(C\ 0 + \ldots + C\ (n) + (n+1))]\,\mathbf{1}$

$T_{1.42} = \mathbb{M}\,0\,(\alpha\ (n+1))$

$T_{1.43} = \mathbb{M}(C\ 0 + \ldots + C\ (n) + (n+1))\,(\alpha\ (n+1))$

$T_{1.431} = \mathbb{M}(C\ 0 + \ldots + C\ (n) + (n+1) + 2)\,(\alpha\ (n+1))$

$T_{1.44} = \mathbb{M}(C\ 0 + \ldots + C\ (n-1) + n + 2)\,(\alpha\ n)$

$T_{1.45} = \mathbb{M}(C\ n+1)\,(\alpha\ (n+1))$

$TI = \alpha; n, C$

D3.1:

$$\frac{}{TI; .; f_u : T_{1.31}; a : \alpha\ n, c : [(C\ n)]\,\mathbf{1} \vdash f_u\ []\ \uparrow^1\langle\!\langle a, c\rangle\!\rangle : \mathbb{M}\,1\,\alpha\ (n+1)}$$

D3:

$$\frac{\dfrac{}{TI; .; f_u : T_{1.31}; . \vdash \mathsf{store}() : \mathbb{M}(C\ n)\,[(C\ n)]\,\mathbf{1}} \qquad D3.1}{TI; .; f_u : T_{1.31}; a : \alpha\ n \vdash \mathsf{bind}\, c = \mathsf{store}()\ \mathsf{in}\ f_u\ []\ \uparrow^1\langle\!\langle a, c\rangle\!\rangle : T_{1.45}}$$

D2.3:

$$\frac{\dfrac{\dfrac{}{TI; .; f_u : T_{1.31}; y_1 : T_{1.411}, b : [n * C]\,\mathbf{1}, b_1 : T_{0.16} \vdash b_1 : T_{0.16}}}{\begin{array}{c}TI; .; f_u : T_{1.31}; y_1 : T_{1.411}, b : [(C\ 0 + \ldots + C\ (n-1) + (n))]\,\mathbf{1}, b_1 : T_{0.16} \vdash \\ \langle\!\langle y_1, b\rangle\!\rangle : (T_{1.411} \otimes [(C\ 0 + \ldots + C\ (n-1) + (n))]\,\mathbf{1})\end{array}}}{\begin{array}{c}TI; .; f_u : T_{1.31}; y_1 : T_{1.411}, b : [(C\ 0 + \ldots + C\ (n-1) + (n))]\,\mathbf{1}, b_1 : T_{0.16} \vdash \\ b_1\ \uparrow^1\langle\!\langle y_1, b\rangle\!\rangle : \mathbb{M}\,1\,\alpha\ n\end{array}}$$

D2.2

$$\frac{}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11} \vdash \overline{N}\ []\ []\ \uparrow^1 ! f_u : T_{0.151}}$$

D2.1:

$$\frac{D2.2 \qquad D2.3}{\begin{array}{c}TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, y_1 : T_{1.411}, b : [(C\ 0 + \ldots + C\ (n-1) + (n))]\,\mathbf{1} \vdash \\ \mathsf{bind}\, b_1 = (\overline{N}\ []\ []\ \uparrow^1 ! f_u)\ \mathsf{in}\ b_1\ \uparrow^1\langle\!\langle y_1, b\rangle\!\rangle : \mathbb{M}\,2\,\alpha\ n\end{array}}$$

D2:

$$\frac{}{TI; .; f_u : T_{1.31}; . \vdash \mathsf{store}() : \mathbb{M}(C\ 0 + \ldots + C\ (n-1) + (n))\,[(C\ 0 + \ldots + C\ (n-1) + (n))]\,\mathbf{1}}$$

$$\frac{D2.1}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, y_1 : T_{1.411} \vdash \mathsf{bind}\, b = \mathsf{store}() \text{ in } \mathsf{bind}\, b_1 = (\overline{N}\ []\ []\ \uparrow^1 !f_u) \text{ in } b_1\ \uparrow^1 \langle\!\langle y_1, b \rangle\!\rangle : T_{1.44}}$$

D1.5:

$$\frac{\dfrac{D2}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, y_1 : T_{1.411} \vdash E_1 : T_{1.44}} \qquad \dfrac{D3}{TI; .; f_u : T_{1.31}; a : \alpha\ n \vdash E_2 : T_{1.45}}}{TI; .; f_u : T_{1.31}; y_1 : T_{1.411} \vdash \mathsf{bind}\, a = E_1 \text{ in } E_2 : T_{1.431}}$$

D1.4:

$$\frac{}{TI; .; f_u : T_{1.31}; p : T_p \vdash p : T_p}$$

D1.3

$$\frac{\dfrac{D1.4 \qquad D1.5}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, p : T_p, y_1 : T_{1.411} \vdash \mathsf{release} - = p \text{ in } \mathsf{bind}\, a = E_1 \text{ in } E_2 : T_{1.43}}}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, p : T_p, y_1 : T_{1.411} \vdash E_0 : T_{1.43}}$$

D1.2

$$\frac{\dfrac{}{TI; .; f_u : T_{1.31}; y_2 : T_{1.412} \vdash y_2 : T_{1.412}} \qquad D1.3}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, p : T_p, y_1 : T_{1.411}, y_2 : T_{1.412} \vdash \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.42}}$$

D1.1

$$\frac{}{TI; .; f_u : T_{1.31}; x : T_{1.41} \vdash x : T_{1.41}}$$

D1:

$$\frac{D1.1 \qquad D1.2}{TI; .; f_u : T_{1.31}; \overline{N} : T_{0.11}, p : T_p, x : T_{1.41} \vdash \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.42}}$$

D0:

$$\frac{}{TI; .; .; f : T_{1.3} \vdash f : T_{1.3}}$$

D0.0:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D0 \qquad D1}{TI; .; .; \overline{N} : T_{0.11}, p : T_p, f : T_{1.31}, x : T_{1.41} \vdash \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.42}}}{TI; .; .; \overline{N} : T_{0.11}, p : T_p, f : T_{1.31} \vdash \lambda x.\, \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.4}}}{TI; .; .; \overline{N} : T_{0.11}, p : T_p, f : T_{1.31} \vdash \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.40}}}{TI; .; .; \overline{N} : T_{0.11}, p : T_p \vdash \lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{1.2}}}{.; n; .; .; \overline{N} : T_{0.11}, p : T_p \vdash \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_1}}{.; n; .; .; \overline{N} : T_{0.11}, p : T_p \vdash \mathsf{ret}\, \Lambda.\Lambda.\lambda f.\, \mathsf{ret}\, \lambda x.\, \mathsf{let}\,!\, f_u = f \text{ in } \mathsf{let}\langle\!\langle y_1, y_2 \rangle\!\rangle = x \text{ in } \mathsf{release} - = y_2 \text{ in } E_0 : T_{0.2}}$$

Main derivation:

$$\frac{\dfrac{\dfrac{\dfrac{D0.0}{.;n;.;.;p:T_p \vdash \lambda\overline{N}.\,\mathsf{ret}\,\Lambda.\Lambda.\lambda f.\,\mathsf{ret}\,\lambda x.\,\mathsf{let}\,!\,f_u = f\;\mathsf{in}\;\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = x\;\mathsf{in}\;\mathsf{release}-=y_2\;\mathsf{in}\;E_0 : T_{0.1}}}{.;n;.;.;p:T_p \vdash \mathsf{ret}\,\lambda\overline{N}.\,\mathsf{ret}\,\Lambda.\Lambda.\lambda f.\,\mathsf{ret}\,\lambda x.\,\mathsf{let}\,!\,f_u = f\;\mathsf{in}\;\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = x\;\mathsf{in}\;\mathsf{release}-=y_2\;\mathsf{in}\;E_0 : T_{0.01}}}{.;n;.;.;. \vdash \lambda p.\,\mathsf{ret}\,\lambda\overline{N}.\,\mathsf{ret}\,\Lambda.\Lambda.\lambda f.\,\mathsf{ret}\,\lambda x.\,\mathsf{let}\,!\,f_u = f\;\mathsf{in}\;\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = x\;\mathsf{in}\;\mathsf{release}-=y_2\;\mathsf{in}\;E_0 : T_{0.0}}}{.;.;.;.;. \vdash \Lambda.\lambda p.\,\mathsf{ret}\,\lambda\overline{N}.\,\mathsf{ret}\,\Lambda.\Lambda.\lambda f.\,\mathsf{ret}\,\lambda x.\,\mathsf{let}\,!\,f_u = f\;\mathsf{in}\;\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = x\;\mathsf{in}\;\mathsf{release}-=y_2\;\mathsf{in}\;E_0 : T_0}$$

Type derivation for add

$add : \forall n_1, n_2.\,[(n_1 * 3 + n_1 + 2)]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))))$

$add = \Lambda.\Lambda.\lambda p.\,\mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0$
where
$E_0 = \mathsf{release} - = p\ \mathsf{in}\ \mathsf{bind}\,a = E_1\ \mathsf{in}\ \ E_2$
$E0.1 = \mathsf{release} - = y_2\ \mathsf{in}\ \mathsf{bind}\,b_1 = (\mathsf{bind}\,b_2 = \mathsf{store}\,()\ \mathsf{in}\ succ\ []\ b_2)\ \mathsf{in}\ b_1\ \uparrow^1 y_1$
$E_1 = \overline{N_1}\ []\ []\ \uparrow^1 !(\Lambda.\lambda t.\,\mathsf{let}\langle\!\langle y_1, y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1)$
$E_2 = \mathsf{bind}\,b = \mathsf{store}()\ \mathsf{in}\ a\ \uparrow^1 \langle\!\langle \overline{N_2}, b\rangle\!\rangle$

$T_p = [(n_1 * 3 + n_1 + 2)]\,\mathbf{1}$
$T_0 = \forall n_1, n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))))$
$T_{0.1} = \forall n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))))$
$T_{0.2} = T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))))$
$T_{0.20} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_1+n_2])))$
$T_{0.21} = (\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_1+n_2])))$
$T_{0.3} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2)))$
$T_{0.31} = \mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))$
$T_{0.4} = \mathbb{M}\,1\,(\mathsf{Nat}\ (n_1+n_2))$
$T_{0.40} = \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1+n_2))$
$T_{0.5} = \mathbb{M}(n_1 * 3 + n_1 + 1)\,(\mathsf{Nat}\ (n_1+n_2))$
$T_{0.6} = \mathbb{M}(n_1 * 3 + n_1 + 2)\,(\mathsf{Nat}\ (n_1+n_2))$
$T_1 =$
$\forall \alpha : \mathbb{N} \to Type.\forall C.!(\forall k.((\alpha\ k \otimes [C\ k]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (k+1)))) \multimap$
$\mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n_1-1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (n_1)))$
$a_f = \lambda k.\mathsf{Nat}\ (n_2 + k)$
$T_{1.1} = \forall C.!(\forall k.((a_f\ k \otimes [C\ k]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ (k+1)))) \multimap$
$\mathbb{M}\,0\,((a_f\ 0 \otimes [(C\ 0 + \ldots + C\ (n_1-1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ n_1))$
$T_{1.2} = \forall C.!(\forall k.((\mathsf{Nat}\ (n_2+k)\ \otimes [C\ k]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 + (k+1))))) \multimap$
$\mathbb{M}\,0\,((\mathsf{Nat}\ (n_2+0) \otimes [(C\ 0 + \ldots + C\ (n_1-1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+n_1)))$
$T_{1.21} = !(\forall k.((\mathsf{Nat}\ (n_2+k)\ \otimes [C\ k]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 + (k+1))))) \multimap$
$\mathbb{M}\,0\,((\mathsf{Nat}\ (n_2+0) \otimes [(C\ 0 + \ldots + C\ (n_1-1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+n_1)))[(\lambda_s - .3)/C]$
$T_{1.22} = !(\forall k.((\mathsf{Nat}\ (n_2+k)\ \otimes [3]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 + (k+1)])))$
$T_{1.23} = (\forall k.((\mathsf{Nat}\ (n_2+k)\ \otimes [3]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 + (k+1)])))$
$T_{1.24} = ((\mathsf{Nat}\ (n_2+k)\ \otimes [3]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 + (k+1))))$
$T_{1.241} = (\mathsf{Nat}\ (n_2+k)\ \otimes [3]\,\mathbf{1})$
$T_{1.2411} = (\mathsf{Nat}\ (n_2+k)$
$T_{1.2412} = [3]\,\mathbf{1}$
$T_{1.242} = \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 + (k+1)))$
$T_{1.3} = \mathbb{M}\,0\,((\mathsf{Nat}\ (n_2+0) \otimes [(n_1 * 3 + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+n_1)))$
$T_{1.30} = \mathbb{M}\,1\,((\mathsf{Nat}\ (n_2+0) \otimes [(n_1 * 3 + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+n_1)))$

$T_{1.31} = ((\mathsf{Nat}\ (n_2+0) \otimes [(n_1*3+n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+n_1)))$
$T_2 = \mathsf{Nat}\ n_2$
$T_3 = (\mathsf{Nat}\ (n_2+k) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2+k+1)))$

D3:

$$\frac{}{.;n_1,n_2;.;.;\overline{N_1}:T_1 \vdash \overline{N_1}:T_1}$$

D2.10:

$$\cfrac{\cfrac{D3 \qquad \cfrac{}{.;n_1,n_2;.;.;. \vdash (\lambda_t k.\mathsf{Nat}[n_2+k]) : \mathbb{N} \to Type}}{.;n_1,n_2;.;.;\overline{N_1}:T_1 \vdash \overline{N_1}\ [] : T_{1.1}}}{.;n_1,n_2;.;.;\overline{N_1}:T_1 \vdash \overline{N_1}\ [] : T_{1.2}}$$

D2:

$$\cfrac{D2.10 \qquad \cfrac{}{.;n_1,n_2;.;.;. \vdash (\lambda_s - .3) : \mathbb{N} \to \mathbb{N}}}{.;n_1,n_2;.;.;\overline{N_1}:T_1 \vdash \overline{N_1}\ []\ [] : T_{1.21}}$$

D1.32:

$$\frac{}{.;n_1,n_2,k;.;.;b_2:[2]\,\mathbf{1} \vdash succ\ []\ b_2 : \mathbb{M}\,0\,T_3}$$

D1.31:

$$\cfrac{\cfrac{}{.;n_1,n_2,k;.;.;. \vdash \mathsf{store}() : \mathbb{M}\,2\,[2]\,\mathbf{1}} \qquad D1.32}{.;n_1,n_2,k;.;.;. \vdash (\mathsf{bind}\,b_2 = \mathsf{store}()\ \mathsf{in}\ succ\ []\ b_2\ ) : \mathbb{M}\,2\,T_3}$$

D1.3:

$$\cfrac{D1.31 \qquad \cfrac{}{.;n_1,n_2,k;.;.;y_1:T_{1.2411},b_1:T_3 \vdash b_1\ \uparrow^1 y_1 : \mathbb{M}\,1\,\mathsf{Nat}[n_2+k+1]}}{.;n_1,n_2,k;.;.;y_1:T_{1.2411} \vdash \mathsf{bind}\,b_1 = (\mathsf{bind}\,b_2 = \mathsf{store}\,()\ \mathsf{in}\ succ\ []\ b_2\ )\ \mathsf{in}\ b_1\ \uparrow^1 y_1 : \mathbb{M}(3)\,\mathsf{Nat}[n_2+k+1]}$$

D1.2:

$$\cfrac{\cfrac{}{.;n_1,n_2,k;.;.;y_2:T_{1.2412} \vdash y_2:T_{1.2412}} \qquad D1.3}{\begin{array}{c} .;n_1,n_2,k;.;.;y_1:T_{1.2411},y_2:T_{1.2412} \vdash \\ \mathsf{release} - = y_2\ \mathsf{in}\ \mathsf{bind}\,b_1 = (\mathsf{bind}\,b_2 = \mathsf{store}\,()\ \mathsf{in}\ succ\ []\ b_2)\ \mathsf{in}\ b_1\ \uparrow^1 y_1 : \mathbb{M}\,0\,\mathsf{Nat}[n_2+k+1] \end{array}}$$

D1.1:

$$\cfrac{\cfrac{\cfrac{}{.;n_1,n_2,k;.;.;t:T_{1.241} \vdash t:T_{1.241}} \qquad \cfrac{D1.2}{.;n_1,n_2,k;.;.;y_1:T_{1.2411},y_2:T_{1.2412} \vdash E0.1 : T_{1.242}}}{.;n_1,n_2,k;.;.;t:T_{1.241} \vdash \mathsf{let}\langle\langle y_1,y_2\rangle\rangle = t\ \mathsf{in}\ E0.1 : T_{1.242}}}{.;n_1,n_2,k;.;.;. \vdash \lambda t.\,\mathsf{let}\langle\langle y_1,y_2\rangle\rangle = t\ \mathsf{in}\ E0.1) : T_{1.24}}$$

D1:

$$\cfrac{D2 \qquad \cfrac{\cfrac{D1.1}{.;n_1,n_2;.;.;. \vdash (\Lambda.\lambda t.\,\mathsf{let}\langle\langle y_1,y_2\rangle\rangle = t\ \mathsf{in}\ E0.1) : T_{1.23}}}{.;n_1,n_2;.;.;. \vdash !(\Lambda.\lambda t.\,\mathsf{let}\langle\langle y_1,y_2\rangle\rangle = t\ \mathsf{in}\ E0.1) : T_{1.22}}}{.;n_1,n_2;.;.;\overline{N_1}:T_1 \vdash \overline{N_1}\ []\ []\ \uparrow^1 !(\Lambda.\lambda t.\,\mathsf{let}\langle\langle y_1,y_2\rangle\rangle = t\ \mathsf{in}\ E0.1) : T_{1.30}}$$

D0.1

$$\frac{D1}{.;n_1,n_2;.;.;\overline{N_1}:T_1,\overline{N_2}:T_2 \vdash E_1 : T_{1.30}}$$

D2.1:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_2, a : T_{1.31}, b : [(n_1 * 3 + n_1)]\,\mathbf{1} \vdash a \ \uparrow^1 \langle\!\langle \overline{N_2}, b \rangle\!\rangle : T_{0.4}}$$

D2.0:

$$\frac{\dfrac{}{.; n_1, n_2; .; .; . \vdash \mathsf{store}() : \mathbb{M}(n_1 * 3 + n_1)\,[(n_1 * 3 + n_1)]\,\mathbf{1}} \qquad D2.1}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash \mathsf{bind}\, b = \mathsf{store}() \ \mathsf{in}\ a \ \uparrow^1 \langle\!\langle \overline{N_2}, b \rangle\!\rangle : T_{0.5}}$$

D0.2:

$$\frac{D2.0}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}$$

D0:

$$\frac{D0.1 \qquad D0.2}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{bind}\, a = E_1 \ \mathsf{in}\ \ E_2 : T_{0.6}}$$

D0.0

$$\frac{\dfrac{}{.; n_1, n_2; .; .; p : T_p \vdash p : T_p} \qquad D0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{release} - = p \ \mathsf{in}\ \mathsf{bind}\, a = E_1 \ \mathsf{in}\ \ E_2 : T_{0.40}}$$

Main derivation:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D0.0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \lambda \overline{N_2}.E_0 : T_{0.31}}}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.3}}}{.; n_1, n_2; .; .; p : T_p \vdash \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.21}}}{.; n_1, n_2; .; .; p : T_p \vdash \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.20}}}{.; n_1, n_2; .; .; . \vdash \lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.2}}}{.; n_1; .; .; . \vdash \Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.1}}}{.; .; .; .; . \vdash \Lambda.\Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_0}$$

<u>Type derivation for *mult*</u>

$mult : \forall n_1, n_2.$
$[(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2)]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2))))$

$mult = \Lambda.\Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}(\lambda \overline{N_2}.E_0)$
where
$E_0 = \mathsf{release} - = p \ \mathsf{in}\ \mathsf{bind}\, a = E_1 \ \mathsf{in}\ \ E_2$
$E0.1 = \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,() \ \mathsf{in}\ add\ []\ []\ b_2 \ \uparrow^1 \overline{N_2}) \ \mathsf{in}\ b_1 \ \uparrow^1 y_1$
$E_1 = \overline{N_1}\ []\ []\ \uparrow^1\ !(\Lambda.\lambda t.\, \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1)$
$E_2 = \mathsf{bind}\, b = \mathsf{store}() \ \mathsf{in}\ a \ \uparrow^1 \langle\!\langle \overline{0}, b \rangle\!\rangle$

$T_p = [(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2)]\,\mathbf{1}$
$T_0 = \forall n_1, n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2))))$
$T_{0.1} = \forall n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2))))$
$T_{0.2} = T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2))))$
$T_{0.21} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_1 * n_2])))$

$T_{0.22} = (\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2))))$

$T_{0.3} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2)))$

$T_{0.31} = (\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1 * n_2)))$

$T_{0.4} = \mathbb{M}\,1\,(\mathsf{Nat}\ (n_1 * n_2))$

$T_{0.5} = \mathbb{M}(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 1)\,(\mathsf{Nat}\ (n_1 * n_2))$

$T_{0.6} = \mathbb{M}(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2)\,(\mathsf{Nat}\ (n_1 * n_2))$

$T_1 =$
$\forall \alpha : \mathbb{N} \to Type.\forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n + 1)))) \multimap$
$\mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n_1 - 1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ n_1))$

$a_f = \lambda k.\mathsf{Nat}[n_2 * k]$

$T_{1.1} = \forall C.!(\forall j_n.((a_f\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ (j_n + 1)))) \multimap$
$\mathbb{M}\,0\,((a_f\ 0 \otimes [(C\ 0 + \ldots + C\ (n_1 - 1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ n_1))$

$T_{1.2} = \forall C.!(\forall j_n.((\mathsf{Nat}[n_2 * j_n]\ \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (j_n + 1)]))) \multimap$
$\mathbb{M}\,0\,((\mathsf{Nat}[n_2 * 0] \otimes [(C\ 0 + \ldots + C\ (n_1 - 1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 * n_1)))$

$T_{1.21} = !(\forall j_n.((\mathsf{Nat}[n_2 * j_n]\ \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (j_n + 1)]))) \multimap$
$\mathbb{M}\,0\,((\mathsf{Nat}[n_2 * 0] \otimes [(C\ 0 + \ldots + C\ (n_1 - 1) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 * n_1)))[C/(\lambda_.(n_2 * 3 + n_2 + 4))]$

$T_{1.22} = !(\forall j_n.((\mathsf{Nat}[n_2 * j_n]\ \otimes [(n_2 * 3 + n_2 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (j_n + 1)])))$

$T_{1.23} = (\forall j_n.((\mathsf{Nat}[n_2 * j_n]\ \otimes [(n_2 * 3 + n_2 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (j_n + 1)])))$

$T_{1.24} = ((\mathsf{Nat}[n_2 * k]\ \otimes [(n_2 * 3 + n_2 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (k + 1)]))$

$T_{1.241} = (\mathsf{Nat}[n_2 * k]\ \otimes [(n_2 * 3 + n_2 + 4)]\,\mathbf{1})$

$T_{1.2411} = (\mathsf{Nat}[n_2 * k]$

$T_{1.2412} = [(n_2 * 3 + n_2 + 4)]\,\mathbf{1}$

$T_{1.242} = \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (k + 1)])$

$T_{1.3} = \mathbb{M}\,0\,((\mathsf{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 * n_1)))$

$T_{1.30} = \mathbb{M}\,1\,((\mathsf{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 * n_1)))$

$T_{1.31} = ((\mathsf{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_2 * n_1)))$

$T_2 = \mathsf{Nat}\ n_2$

$T_3 = (\mathsf{Nat}[n_2 * k] \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2 * (k + 1)]))$

D3:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} : T_1}$$

D2.10:

$$\frac{\dfrac{D3 \qquad \dfrac{}{.; n_1, n_2; .; .; . \vdash (\lambda_t k.\mathsf{Nat}[n_2 * k]) : \mathbb{N} \to Type}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1}\ [] : T_{1.1}}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1}\ [] : T_{1.2}}$$

D2:

$$\frac{D2.10 \qquad \dfrac{}{.; n_1, n_2; .; .; . \vdash (\lambda_s - .(n_2 * 3 + n_2 + 4)) : S \to S}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1}\ []\ [] : T_{1.21}}\ \text{T-iapp}$$

D1.32

$$\frac{}{.; n_1, n_2, k; .; .; . \vdash add\ []\ []\ b_2 \uparrow^1 \overline{N_2} : \mathbb{M}\,1\,T_3}$$

D1.31

$$\frac{\dfrac{}{.; n_1, n_2, k; .; .; . \vdash \mathsf{store}\,() : \mathbb{M}(n_2 * 3 + n_2 + 2)\,[(n_2 * 3 + n_2 + 2)]\,\mathbf{1}} \qquad D1.32}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash (\mathsf{bind}\, b_2 = \mathsf{store}()\ \mathsf{in}\ add\ []\ []\ b_2 \uparrow^1 \overline{N_2}) : \mathbb{M}(n_2 * 3 + n_2 + 3)\,T_3}$$

D1.3

$$\dfrac{D1.31 \qquad \dfrac{}{.;n_1,n_2,k;.;.;y_1 : T_{1.241}, b_1 : T_3 \vdash b_1 \uparrow^1 y_1 : \mathbb{M}\,1\,\mathsf{Nat}[n_2*(k+1)]}}{\begin{array}{c}.;n_1,n_2,k;.;.;y_1 :\vdash \mathsf{bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,()\ \mathsf{in}\ add\ []\ []\ b_2 \uparrow^1 \overline{N_2})\ \mathsf{in}\ b_1 \uparrow^1 y_1 : \\ \mathbb{M}(n_2*3+n_2+4)\,\mathsf{Nat}[n_2*(k+1)]\end{array}}$$

D1.2:

$$\dfrac{\dfrac{}{.;n_1,n_2,k;.;.;y_2 : T_{1.2412} \vdash y_2 : T_{1.2412}} \qquad D1.3}{\begin{array}{c}.;n_1,n_2,k;.;.;y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash \\ \mathsf{release} - = y_2\ \mathsf{in}\ \mathsf{bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,()\ \mathsf{in}\ add\ []\ []\ b_2 \uparrow^1 \overline{N_2})\ \mathsf{in}\ b_1 \uparrow^1 y_1 : \mathbb{M}\,0\,\mathsf{Nat}[n_2*(k+1)]\end{array}}$$

D1.1

$$\dfrac{\dfrac{\dfrac{}{.;n_1,n_2,k;.;.;t : T_{1.241} \vdash t : T_{1.241}} \qquad \dfrac{D1.2}{.;n_1,n_2,k;.;.;y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash E0.1 : T_{1.242}}}{.;n_1,n_2,k;.;.;t : T_{1.241} \vdash \mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1) : T_{1.242}}}{.;n_1,n_2,k;.;.;. \vdash \lambda t.\,\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1) : T_{1.24}}$$

D1:

$$\dfrac{D2 \qquad \dfrac{\dfrac{D1.1}{.;n_1,n_2;.;.;. \vdash (\Lambda.\lambda t.\,\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1) : T_{1.23}}}{.;n_1,n_2;.;.;. \vdash !(\Lambda.\lambda t.\,\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1) : T_{1.22}}}{.;n_1,n_2;.;.;\overline{N_1} : T_1 \vdash \overline{N_1}\ []\ []\ \uparrow^1 !(\Lambda.\lambda t.\,\mathsf{let}\langle\!\langle y_1,y_2\rangle\!\rangle = t\ \mathsf{in}\ E0.1) : T_{1.30}}$$

D0.1:

$$\dfrac{D1}{.;n_1,n_2;.;.;\overline{N_1} : T_1, \overline{N_2} : T_2 \vdash E_1 : T_{1.30}}$$

D2.1:

$$\dfrac{}{.;n_1,n_2;.;.;\overline{N_2} : T_2, a : T_{1.31}, b : [(n_1*(n_2*3+n_2+4)+n_1)]\,\mathbf{1} \vdash a \uparrow^1 \langle\!\langle \overline{0}, b\rangle\!\rangle : T_{0.4}}$$

D2.0:

$$\dfrac{\dfrac{}{.;n_1,n_2;.;.;. \vdash \mathsf{store}() : \mathbb{M}(n_1*(n_2*3+n_2+4)+n_1)\,[(n_1*(n_2*3+n_2+4)+n_1)]\,\mathbf{1}} \quad D2.1}{.;n_1,n_2;.;.;\overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash \mathsf{bind}\, b = \mathsf{store}()\ \mathsf{in}\ a \uparrow^1 \langle\!\langle \overline{0}, b\rangle\!\rangle : T_{0.5}}$$

D0.2:

$$\dfrac{D2.0}{.;n_1,n_2;.;.;\overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}$$

D0:

$$\dfrac{D0.1 \qquad D0.2}{.;n_1,n_2;.;.;\overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{bind}\, a = E_1\ \mathsf{in}\ \ E_2 : T_{0.6}}$$

D0.0

$$\dfrac{\dfrac{}{.;n_1,n_2;.;.;p : T_p \vdash p : T_p} \qquad D0}{.;n_1,n_2;.;.;p : T_p, \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{release} - = p\ \mathsf{in}\ \mathsf{bind}\, a = E_1\ \mathsf{in}\ \ E_2 : T_{0.4}}$$

Main derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{D0.0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \lambda\overline{N_2}.E_0 : T_{0.31}}}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_{0.3}}}{.; n_1, n_2; .; .; p : T_p \vdash \lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_{0.22}}}{.; n_1, n_2; .; .; p : T_p \vdash \mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_{0.21}}}{.; n_1, n_2; .; .; . \vdash \lambda p.\,\mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_{0.2}}}{.; n_1; .; .; . \vdash \Lambda.\lambda p.\,\mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_{0.1}}}{.; .; .; .; . \vdash \Lambda.\Lambda.\lambda p.\,\mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0 : T_0}$$

Type derivation for exp

$exp : \forall n_1, n_2.\,[\sum_{i\in\{0,n_2-1\}}(\lambda k.(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)\ (i)) + n_2 + 2]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2}))))$

$exp = \Lambda.\Lambda.\lambda p.\,\mathsf{ret}\,\lambda\overline{N_1}.\,\mathsf{ret}\,\lambda\overline{N_2}.E_0$

where

$E_0 = \mathsf{release} - = p \mathsf{\ in\ bind}\, a = E_1 \mathsf{\ in\ }\ E_2$

$E_{0.1} = \mathsf{release} - = y_2 \mathsf{\ in\ bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,() \mathsf{\ in\ } mult\ []\ []\ b_2\ \uparrow^1 \overline{N_1}) \mathsf{\ in\ } b_1\ \uparrow^1 y_1$

$E_1 = \overline{N_2}\ []\ []\ \uparrow^1\ !(\Lambda.\lambda t.\,\mathsf{let}\langle\langle y_1, y_2\rangle\rangle = t \mathsf{\ in\ } E0.1)$

$E_2 = \mathsf{bind}\, b = \mathsf{store}\,\mathbf{1} \mathsf{\ in\ } a\ \uparrow^1\langle\langle\overline{1}, b\rangle\rangle$

$P = \sum_{i\in\{0,n_2-1\}}(\lambda k.(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)\ i) + n_2 + 2$

$T_p = [P]\,\mathbf{1}$

$T_b = [P-1]\,\mathbf{1}$

$T_0 = \forall n_1, n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2}))))$

$T_{0.1} = \forall n_2.T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2}))))$

$T_{0.2} = T_p \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2}))))$

$T_{0.20} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_1^{n_2}])))$

$T_{0.21} = \mathsf{Nat}\ n_1 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{0.3} = \mathbb{M}\,0\,(\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{0.31} = (\mathsf{Nat}\ n_2 \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{0.4} = \mathbb{M}\,1\,(\mathsf{Nat}\ (n_1^{n_2}))$

$T_{0.5} = \mathbb{M}(P-1)\,(\mathsf{Nat}\ (n_1^{n_2}))$

$T_{0.6} = \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2}))$

$T_1 =$

$\forall\alpha : \mathbb{N} \to Type.\forall C.!(\forall j_n.((\alpha\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ (j_n+1)))) \multimap \mathbb{M}\,0\,((\alpha\ 0 \otimes [(C\ 0 + \ldots + C\ (n_2-1) + n_2)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\alpha\ n_2))$

$a_f = \lambda k.\mathsf{Nat}[n_2^k]$

$T_{1.1} =$

$\forall C.!(\forall j_n.((a_f\ j_n \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ (j_n+1)))) \multimap \mathbb{M}\,0\,((a_f\ 0 \otimes [(C\ 0 + \ldots + C\ (n_2-1) + n_2)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(a_f\ n_2))$

$T_{1.2} =$

$\forall C.!(\forall j_n.((\mathsf{Nat}[n_2^{j_n}]\ \otimes [C\ j_n]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(j_n+1)}]))) \multimap \mathbb{M}\,0\,((\mathsf{Nat}[n_2^0] \otimes [(C\ 0 + \ldots + C\ (n_2-1) + n_2)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{1.21} =$

$!(\forall j_n.((\mathsf{Nat}[n_2^{j_n}]\ \otimes [(n_1 * (n_1^{j_n} * 3 + n_1^{j_n} + 4) + n_1 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(j_n+1)}]))) \multimap \mathbb{M}\,0\,((\mathsf{Nat}[n_2^0] \otimes [P]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$P =$

$(\lambda k.(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4))\ 0 + \ldots + (\lambda k.(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4))\ (n_2 - 1) + n_2$

$T_{1.22} = !(\forall k.((\mathsf{Nat}[n_2^k]\ \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(k+1)}])))$

$T_{1.23} = (\forall k.((\mathsf{Nat}[n_2^k]\ \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(k+1)}])))$

$T_{1.24} = ((\mathsf{Nat}[n_2^k]\ \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(k+1)}]))$

$T_{1.241} = (\mathsf{Nat}[n_2^k]\ \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)]\,\mathbf{1})$

$T_{1.2411} = \mathsf{Nat}[n_2^k]$

$T_{1.2412} = ([(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)]\,\mathbf{1})$

$T_{1.242} = \mathbb{M}\,0\,(\mathsf{Nat}[n_2^{(k+1)}])$

$T_{1.3} = \mathbb{M}\,0\,((\mathsf{Nat}[n_2^0] \otimes [P]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{1.30} = \mathbb{M}\,1\,((\mathsf{Nat}[n_2^0] \otimes [P]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_{1.31} = ((\mathsf{Nat}[n_2^0] \otimes [P]\,\mathbf{1}) \multimap \mathbb{M}\,0\,(\mathsf{Nat}\ (n_1^{n_2})))$

$T_2 = \mathsf{Nat}\ n_1$

$T_3 = (\mathsf{Nat}[n_1^k] \multimap \mathbb{M}\,0\,(\mathsf{Nat}[n_1^{(k+1)}]))$

D3:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2} : T_1}$$

D2.1:

$$\frac{\dfrac{D3 \qquad \dfrac{}{.; n_1, n_2; .; .; . \vdash (\lambda_t k.\mathsf{Nat}[n_2^k]) : \mathbb{N} \to Type}}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2}\ [] : T_{1.1}}}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2}\ [] : T_{1.2}}$$

D2:

$$\frac{D2.1 \qquad \dfrac{}{.; n_1, n_2; .; .; . \vdash (\lambda_s k.(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2)) : \mathbb{N} \to \mathbb{N}}}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2}\ []\ [] : T_{1.21}}$$

D1.32

$$\frac{}{.; n_1, n_2, k; .; .; b_2 : [((n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2))]\,\mathbf{1} \vdash mult\ []\ []\ b_2\ \uparrow^1 \overline{N_1} : \mathbb{M}\,1\,T_3}$$

D1.31

$$\frac{\dfrac{}{.; n_1, n_2, k; .; .; . \vdash \mathsf{store}\,() : \mathbb{M}((n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2))\,[((n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2))]\,\mathbf{1}} \quad D1.32}{\begin{array}{c} .; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash (\mathsf{bind}\, b_2 = \mathsf{store}\,()\ \mathsf{in}\ mult\ []\ []\ b_2\ \uparrow^1 \overline{N_1}) : \\ \mathbb{M}(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 3)\,T_3 \end{array}}$$

D1.3

$$\frac{D1.31 \qquad \dfrac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash b_1\ \uparrow^1 y_1 : \mathbb{M}\,1\,\mathsf{Nat}[n_2^{(k+1)}]}}{\begin{array}{c} .; n_1, n_2, k; .; .; y_1 :\vdash \mathsf{bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,()\ \mathsf{in}\ mult\ []\ []\ b_2\ \uparrow^1 \overline{N_1})\ \mathsf{in}\ b_1\ \uparrow^1 y_1 : \\ \mathbb{M}(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)\,\mathsf{Nat}[n_2^{(k+1)}] \end{array}}$$

D1.2:

$$\dfrac{\dfrac{}{.; n_1, n_2, k; .; .; y_2 : T_{1.2412} \vdash y_2 : T_{1.2412}} \qquad D1.3}{\begin{array}{c} .; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash \\ \mathsf{release} - = y_2 \ \mathsf{in}\ \mathsf{bind}\, b_1 = (\mathsf{bind}\, b_2 = \mathsf{store}\,()\ \mathsf{in}\ mult\ b_2\ n_1\ (n_1^k) \uparrow^1 \overline{N_1})\ \mathsf{in}\ b_1 \uparrow^1 y_1 : \mathbb{M}\,0\,\mathsf{Nat}[n_2^{(k+1)}] \end{array}}$$

D1.1

$$\dfrac{\dfrac{\dfrac{}{.; n_1, n_2, k; .; .; t : T_{1.241} \vdash t : T_{1.241}} \qquad \dfrac{D1.2}{.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash E0.1 : T_{1.242}}}{.; n_1, n_2, k; .; .; t : T_{1.241} \vdash \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1) : T_{1.242}}}{.; n_1, n_2, k; .; .; . \vdash \lambda t.\, \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1) : T_{1.24}}$$

D1:

$$\dfrac{D2 \qquad \dfrac{\dfrac{D1.1}{.; n_1, n_2; .; .; . \vdash (\Lambda.\lambda t.\, \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1) : T_{1.23}}}{.; n_1, n_2; .; .; . \vdash !(\Lambda.\lambda t.\, \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1) : T_{1.22}}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1}\ []\ []\ \uparrow^1 !(\Lambda.\lambda t.\, \mathsf{let} \langle\!\langle y_1, y_2 \rangle\!\rangle = t \ \mathsf{in}\ E0.1) : T_{1.30}}$$

D0.1:

$$\dfrac{D1}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash E_1 : T_{1.30}}$$

D2.1:

$$\dfrac{}{.; n_1, n_2; .; .; \overline{N_2} : T_2, a : T_{1.31}, b : T_b \vdash a \uparrow^1 \langle\!\langle \overline{1}, b \rangle\!\rangle : T_{0.4}}$$

D2.0:

$$\dfrac{\dfrac{}{.; n_1, n_2; .; .; . \vdash \mathsf{store}() : \mathbb{M} (P-2)\, T_b} \qquad D2.1}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash \mathsf{bind}\, b = \mathsf{store}()\ \mathsf{in}\ a \uparrow^1 \langle\!\langle \overline{1}, b \rangle\!\rangle : T_{0.5}}$$

D0.2:

$$\dfrac{D2.0}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}$$

D0:

$$\dfrac{D0.1 \qquad D0.2}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{bind}\, a = E_1 \ \mathsf{in}\ E_2 : T_{0.5}}$$

D0.0

$$\dfrac{\dfrac{}{.; n_1, n_2; .; .; p : T_p \vdash p : T_p} \qquad D0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \mathsf{release} - = p \ \mathsf{in}\ \mathsf{bind}\, a = E_1 \ \mathsf{in}\ E_2 : T_{0.6}}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D0.0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \lambda \overline{N_2}.E_0 : T_{0.31}}}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.3}}}{.; n_1, n_2; .; .; p : T_p \vdash \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.21}}}{.; n_1, n_2; .; .; p : T_p \vdash \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.20}}}{.; n_1, n_2; .; .; . \vdash \lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.2}}}{.; n_1; .; .; . \vdash \Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_{0.1}}}{.; .; .; .; . \vdash \Lambda.\Lambda.\lambda p.\, \mathsf{ret}\, \lambda \overline{N_1}.\, \mathsf{ret}\, \lambda \overline{N_2}.E_0 : T_0}$$

3.3 Fold

$$\frac{}{\Psi; \Theta; \Delta; .; . \vdash 0 : \mathsf{Nat}(0)} \text{ T-nat} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash n : \mathsf{Nat}(n)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash n-1 : \mathsf{Nat}(n-1)} \text{ T-sub}$$

$foldr : \forall \alpha, \beta, n, C : \mathbb{N} \to \mathbb{R}^+.$
$!(\forall i. [Ci]\, \mathbf{1} \multimap \mathsf{Nat}(i) \multimap \alpha \multimap \beta \multimap \mathbb{M}\, 0\, \beta) \multimap !\mathsf{Nat}(n) \multimap \beta \multimap L^n \alpha \multimap [\sum_{i<n} C\ i]\, \mathbf{1} \multimap \mathbb{M}\, 0\, \beta$

$foldr \triangleq \mathsf{fix} f'.\Lambda.\Lambda.\Lambda.\Lambda.\lambda f\ \ c\ \ s\ \ ls\ \ p.$
$\mathsf{let}\ \ !f_u = f\ \ \mathsf{in}$
$\quad \mathsf{let}\ \ !c_u = c\ \ \mathsf{in}$
$\quad\quad \mathsf{match}\ \ ls\ \ \mathsf{with}$
$\quad\quad\quad |nil \mapsto \mathsf{ret}\, s$
$\quad\quad\quad |h :: t \mapsto \mathsf{release}\, \_ = p\ \mathsf{in}$
$\quad\quad\quad\quad\quad\quad \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\quad\quad\quad\quad\quad \mathsf{bind}\, p'' = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\quad\quad\quad\quad\quad\quad \mathsf{bind}\, tr = f'\ \ [][][][]\ \ !f_u\ \ !(c_u - 1)\ \ s\ \ t\ \ p''\ \mathsf{in}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad (f_u\ \ []\ \ p'\ \ (c_u - 1)\ \ h\ \ tr)$

Listing 1: fold function

$E_0 = \mathsf{fix} f'.E_1$
$E_1 = \Lambda.\Lambda.\Lambda.\Lambda.E_2$
$E_2 = \lambda f\ c\ s\ ls\ p.E_3$
$E_3 = \mathsf{let}\ !f_u = f\ \mathsf{in} E_{4.0}$
$E_{4.0} = \mathsf{let}\ !n_u = n\ \mathsf{in} E_4$
$E_4 = \mathsf{match}\ ls\ \mathsf{with} |nil \mapsto \mathsf{ret}\, s | h :: t \mapsto E_5$
$E_5 = \mathsf{release}\, \_ = p\ \mathsf{in}\ E_6$
$E_6 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_7$
$E_7 = \mathsf{bind}\, p'' = \mathsf{store}()\ \mathsf{in}\ E_8$
$E_8 = \mathsf{bind}\, tr = f'\ [][][][]\ !f_u\ !(c_u - 1)\ s\ t\ p''\ \mathsf{in}\ E_9$
$E_9 = (f\ []\ p'\ (c_u - 1)\ h\ tr)$

$T_0 = \forall \alpha, \beta, n, C : \mathbb{N} \to \mathbb{R}^+.T_1$
$T_1 = !T_2 \multimap !T_{2.1} \multimap \beta \multimap T_3 \multimap T_4 \multimap T_5$
$T_2 = \forall i. [C\ i]\, \mathbf{1} \multimap \mathsf{Nat}(i) \multimap \alpha \multimap \beta \multimap \mathbb{M}\, 0\, \beta$
$T_{2.0} = [C\ (n-1)]\, \mathbf{1} \multimap \mathsf{Nat}(n-1) \multimap \alpha \multimap \beta \multimap \mathbb{M}\, 0\, \beta$
$T_{2.01} = \mathsf{Nat}(n-1) \multimap \alpha \multimap \beta \multimap \mathbb{M}\, 0\, \beta$
$T_{2.02} = \alpha \multimap \beta \multimap \mathbb{M}\, 0\, \beta$
$T_{2.03} = \beta \multimap \mathbb{M}\, 0\, \beta$
$T_{2.1} = \mathsf{Nat}(n)$
$T_3 = L^n \alpha$
$T_4 = [\sum_{i<n} C\ i]\, \mathbf{1}$
$T_{4.0} = \mathbb{M}(\sum_{i<n} C\ i)\, [(\sum_{i<n} C\ i)]\, \beta$
$T_{4.1} = [(C(n-1))]\, \mathbf{1}$
$T_{4.10} = \mathbb{M}(C(n-1))\, [(C(n-1))]\, \mathbf{1}$
$T_{4.2} = [\sum_{i<n-1} C\ i]\, \mathbf{1}$
$T_{4.20} = \mathbb{M}(\sum_{i<n-1} C\ i)\, [(\sum_{i<n-1} C\ i)]\, \mathbf{1}$
$T_5 = \mathbb{M}\, 0\, \beta$
$T_6 = \mathbb{M}(\sum_{i<n-1} C\ i)\, \beta$
$T_6 = \mathbb{M}(\sum_{i<n} C\ i)\, \beta$

D6.5:

$$\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash f_u : T_2}$$

D6.4:

$$\frac{D6.5 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0 \vdash n - 1 : \mathbb{N}}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash f_u \; [] : T_{2.0}}$$

D6.3:

$$\frac{D6.4 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash p' : T_{4.1}}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash f_u \; [] \; p' : T_{2.01}}$$

D6.2:

$$\frac{D6.3 \qquad \frac{\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash c_u : T_{2.1}}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash (c_u - 1) : \mathsf{Nat}(n-1)}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash f_u \; [] \; p' \; (c_u - 1) : T_{2.02}}$$

D6.1:

$$\frac{D6.2 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash h : \alpha}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1} \vdash f_u \; [] \; p' \; (c_u - 1) \; h : T_{2.03}}$$

D6:

$$\frac{\frac{D6.1 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; tr : \beta \vdash tr : \beta}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash (f_u \; [] \; p' \; (c_u - 1) \; h \; tr) : T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash E_9 : T_5}$$

D5.5:

$$\frac{\frac{}{\begin{array}{c}\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash f' \; [] [] : \\ \forall n, C : \mathbb{N} \to \mathbb{R}^+ . !T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5\end{array}} \qquad \frac{}{\alpha, \beta, n, C; n; n > 0 \vdash n - 1 : \mathbb{N}}}{\begin{array}{c}\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash f' \; [] [] [] : \\ \forall C : \mathbb{N} \to \mathbb{R}^+ . !T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5\end{array}}$$

D5.4:

$$\frac{D5.5 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0 \vdash C : \mathbb{N} \to \mathbb{R}^+}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash f' \; [] [] [] [] : !T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.3:

$$\frac{D5.4 \qquad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash !f_u : !T_2}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; . \vdash f' \; [] [] [] [] \; !f_u : !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.21:

$$\dfrac{D5.3 \qquad \dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;.\vdash !(c_u-1) : !T_{2.1}}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;.\vdash f' \ \square\square\square\square\ !f_u\ !(c_u-1) : \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.2:

$$\dfrac{D5.21 \qquad \dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta\vdash s:\beta}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta\vdash f' \ \square\square\square\square\ !f_u\ !(c_u-1)\ s : L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.1:

$$\dfrac{D5.2 \qquad \dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;t:L^{n-1}\alpha\vdash t:L^{n-1}\alpha}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,t:L^{n-1}\alpha\vdash f' \ \square\square\square\square\ !f_u\ !(c_u-1)\ s\ t : T_{4.2} \multimap T_5}$$

D5:

$$\dfrac{\dfrac{D5.1 \qquad \dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;p'':T_{4.2}\vdash p'':T_{4.2}}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,t:L^{n-1}\alpha,p'':T_{4.2}\vdash f' \ \square\square\square\square\ !f_u\ !(c_u-1)\ s\ t\ p'' : T_5} \quad D6}{\dfrac{\begin{array}{c}\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p':T_{4.1},p'':T_{4.2}\vdash \\ \mathsf{bind}\, tr = f' \ \square\square\square\square\ !f_u\ !(c_u-1)\ s\ t\ p''\ \mathsf{in}\ E_9 : T_5\end{array}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p':T_{4.1},p'':T_{4.2}\vdash E_8:T_5}}$$

D4.1:

$$\dfrac{\dfrac{\dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;.\vdash \mathsf{store}() : T_{4.20}} \qquad E_5}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p':T_{4.1}\vdash \mathsf{bind}\, p''=\mathsf{store}()\ \mathsf{in}\ E_8 : T_6}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p':T_{4.1}\vdash E_7:T_6}$$

D4:

$$\dfrac{\dfrac{\dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;.\vdash \mathsf{store}() : T_{4.10}} \qquad D4.1}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha\vdash \mathsf{bind}\, p'=\mathsf{store}()\ \mathsf{in}\ E_7 : T_7}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha\vdash E_6:T_7}$$

D3:

$$\dfrac{\dfrac{\dfrac{}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;p:T_4\vdash p:T_4} \qquad D4}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p:T_4\vdash \mathsf{release}\ \_ = p\ \mathsf{in}\ E_6 : T_5}}{\alpha,\beta,n,C;n;n>0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,h:\alpha,t:L^{n-1}\alpha,p:T_4\vdash E_5:T_5}$$

D2:

$$\dfrac{}{\alpha,\beta,n,C;n;n=0;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,p:T_4\vdash \mathsf{ret}\, s : T_5}$$

D1:

$$\dfrac{\dfrac{\dfrac{}{\alpha,\beta,n,C;n;.;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,ls:T_3,p:T_4\vdash ls:T_3} \qquad D2 \qquad D3}{\alpha,\beta,n,C;n;.;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,ls:T_3,p:T_4\vdash \mathsf{match}\ ls\ \mathsf{with}|nil\mapsto \mathsf{ret}\, s|h::t\mapsto E_5 : T_5}}{\alpha,\beta,n,C;n;.;c_u:T_{2.1},f_u:T_2,f':T_0;s:\beta,ls:T_3,p:T_4\vdash E_4:T_5}$$

D0.0:

$$\dfrac{\dfrac{}{\alpha, \beta, n, C; n; .; f' : T_0; f :!T_2, c :!T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash c :!T_{2.1}} \qquad D1}{\alpha, \beta, n, C; n; .; f_u : T_2, f' : T_0; c :!T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash \mathsf{let}\ !c_u = c\ \mathsf{in} E_4 :}$$

D0:

$$\dfrac{\dfrac{}{\alpha, \beta, n, C; n; .; f' : T_0; f :!T_2, c :!T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash f :!T_2} \qquad D0.0}{\alpha, \beta, n, C; n; .; f' : T_0; f :!T_2, c :!T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash \mathsf{let}\ !f_u = f\ \mathsf{in} E_{4.0} : T_5}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{D0}{\alpha, \beta, n, C; n; .; f' : T_0; f :!T_2, c :!T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash E_3 : T_5}}{\alpha, \beta, n, C; .; .; f' : T_0; . \vdash E_2 : T_1}}{; .; .; f' : T_0; . \vdash E_1 : T_0}}{.; .; .; .; . \vdash E_0 : T_0}$$

3.4 Append

$append : \forall s_1, s_2. L^{s_1}[1]\,\tau \multimap L^{s_2}\tau \multimap \mathbb{M}\,0\,(L^{s_1+s_2}\tau)$
$append \triangleq \mathsf{fix} f.\Lambda.\Lambda.\lambda l_1 l_2.E_0$
$E_0 = \mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_{0.1}\ |h :: t \mapsto E_{0.2}$
$E_{0.1} = \mathsf{ret}\, nil :: l_2$
$E_{0.2} = \mathsf{release}\, h_e = h\ \mathsf{in}\ \mathsf{bind}\, t_e = f[][]\ t\ l_2\ \mathsf{in}\ E_{0.3}$
$E_{0.3} = \mathsf{bind} - = \uparrow^1\ \mathsf{in}\ \mathsf{ret}\, h_e :: t_e$

Typing derivation

$E_0 = \mathsf{match}\ l_1\ \mathsf{with}\ |nil \mapsto E_{0.1}\ |h :: t \mapsto E_{0.2}$
$E_{0.1} = \mathsf{ret}\, nil :: l_2$
$E_{0.2} = \mathsf{release}\, h_e = h\ \mathsf{in}\ \mathsf{bind}\, t_e = f[][]\ t\ l_2\ \mathsf{in}\ E_{0.3}$
$E_{0.3} = \mathsf{bind} - = \uparrow^1\ \mathsf{in}\ \mathsf{ret}\, h_e :: t_e$

$T_0 = \forall s_1, s_2. L^{s_1}[1]\,\tau \multimap L^{s_2}\tau \multimap \mathbb{M}\,0\,(L^{s_1+s_2}\tau)$
$T_1 = L^{s_1}[1]\,\tau \multimap L^{s_2}\tau \multimap \mathbb{M}\,0\,(L^{s_1+s_2}\tau)$
$T_{1.1} = L^{s_1}[1]\,\tau$
$T_{1.2} = L^{s_2}\tau$
$T_{1.3} = \mathbb{M}\,0\,(L^{s_1+s_2}\tau)$
$T_2 = L^{s_2}\tau \multimap \mathbb{M}\,s_1\,(L^{s_1+s_2}\tau)$

D1.2:

$$\dfrac{\dfrac{}{.; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2}\tau \vdash (h_e :: t_e) : L^{s_1+s_2}\tau}}{.; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2}\tau \vdash \mathsf{ret}(h_e :: t_e) : \mathbb{M}\,0\,(L^{s_1+s_2}\tau)}$$

D1.1:

$$\dfrac{\dfrac{}{.; s_1, s_2; s_1 > 0; . \vdash \uparrow^1 : \mathbb{M}\,1\,\mathbf{1}} \qquad D1.2}{.; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2}\tau \vdash \mathsf{bind} - = \uparrow^1\ \mathsf{in}\ \mathsf{ret}\, h_e :: t_e : \mathbb{M}\,1\,(L^{s_1+s_2}\tau)}$$

D1.0:

$$\dfrac{\dfrac{}{.; s_1, s_2; s_1 > 0; f : T_0; t : L^{s_1-1}\tau, l_2 : L^{s_2}\tau \vdash f[][]\ t\ l_2 : \mathbb{M}(0)\,(L^{s_1-1+s_2}\tau)} \; D1.1}{.; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t : L^{s_1-1}\tau, l_2 : L^{s_2}\tau \vdash \mathsf{bind}\, t_e = f[][]\ t\ l_2 \ \mathsf{in}\ \mathsf{ret}(h_e :: t_e) : \mathbb{M}\,1\,(L^{s_1+s_2}\tau)}$$

D1:

$$\dfrac{\dfrac{}{.; s_1, s_2; s_1 > 0; f : T_0; h : [1]\,\tau \vdash h : [1]\,\tau} \; D1.0}{.; s_1, s_2; s_1 > 0; f : T_0; h : [1]\,\tau, t : L^{s_1-1}\tau, l_2 : T_{1.2} \vdash E_{0.2} : \mathbb{M}\,0\,(L^{(s_1+s_2)}\tau)}$$

D0:

$$\dfrac{\dfrac{}{.; s_1, s_2; s_1 = 0; f : T_0; l_2 : T_{1.2} \vdash l_2 : L^{s_2}\tau}}{.; s_1, s_2; s_1 = 0; f : T_0; l_2 : T_{1.2} \vdash \mathsf{ret}\, l_2 : \mathbb{M}\,0\,(L^{s_1+s_2})\tau}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{.; s_1, s_2; .; f : T_0; l_1 : T_{1.1} \vdash l_1 : T_{1.1}} \quad D0 \quad D1}{.; s_1, s_2; .; f : T_0; l_1 : T_{1.1}, l_2 : T_{1.2} \vdash E_0 : \mathbb{M}\,0\,(L^{s_1+s_2}\tau)}}{.; s_1, s_2; .; f : T_0; . \vdash \lambda l_1 l_2.E_0 : T_1}}{.; .; .; f : T_0; . \vdash \Lambda.\Lambda.\lambda l_1 l_2.E_0 : T_0}}{.; .; .; .; . \vdash \mathsf{fix} f.\Lambda.\Lambda.\lambda l_1 l_2.E_0 : T_0}$$

3.5 Map

$map : \forall n, c.!(\tau_1 \multimap \mathbb{M}\, c\, \tau_2) \multimap L^n([c]\,\tau_1) \multimap \mathbb{M}\,0\,(L^n\tau_2)$
$map \triangleq$
$\mathsf{fix} f.\Lambda.\Lambda.\lambda gl.\mathsf{let}\,!\, g_u = g \ \mathsf{in}\ E_0$
$E_0 = \mathsf{match}\ l\ \mathsf{with}\ |nil \mapsto E_{0.1}\ |h :: t \mapsto E_{0.2}$
$E_{0.1} = \mathsf{ret}\, nil$
$E_{0.2} = \mathsf{release}\, h_e = h\ \mathsf{in}\ E_{0.3}$
$E_{0.3} = \mathsf{bind}\, h_n = g_u\ h_e\ \mathsf{in}\ E_{0.4}$
$E_{0.4} = \mathsf{bind}\, t_n = f[][]\ !g_u\ t\ \mathsf{in}\ \mathsf{ret}\, h_n :: t_n$

Typing derivation

$E = \mathsf{fix} f.\Lambda.\Lambda.\lambda gl.\mathsf{let}\,!\, g_u = g\ \mathsf{in}\ E_0$
$E_0 = \mathsf{match}\ l\ \mathsf{with}\ |nil \mapsto E_{0.1}\ |h :: t \mapsto E_{0.2}$
$E_{0.1} = \mathsf{ret}\, nil$
$E_{0.2} = \mathsf{release}\, h_e = h\ \mathsf{in}\ E_{0.3}$
$E_{0.3} = \mathsf{bind}\, h_n = g_u\ h_e\ \mathsf{in}\ E_{0.4}$
$E_{0.4} = \mathsf{bind}\, t_n = f[][]\ !g_u\ t\ \mathsf{in}\ \mathsf{ret}\, h_n :: t_n$
$E_1 = \Lambda.\Lambda.\lambda gl.\mathsf{let}\,!\, g_u = g\ \mathsf{in}\ E_0$
$E_2 = \lambda gl.\mathsf{let}\,!\, g_u = g\ \mathsf{in}\ E_0$
$E_3 = \mathsf{let}\,!\, g_u = g\ \mathsf{in}\ E_0$

$T_0 = \forall n, c.!(\tau_1 \multimap \mathbb{M}\, c\, \tau_2) \multimap L^n([c]\,\tau_1) \multimap \mathbb{M}\,0\,(L^n\tau_2)$
$T_1 =!(\tau_1 \multimap \mathbb{M}\, c\, \tau_2) \multimap L^n([c]\,\tau_1) \multimap \mathbb{M}\,0\,(L^n\tau_2)$
$T_{1.1} = (\tau_1 \multimap \mathbb{M}\, c\, \tau_2)$
$T_{1.2} = L^n([c]\,\tau_1)$
$T_{1.3} = \mathbb{M}\,0\,(L^n\tau_2)$

D1.2:

$$\frac{}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2, t_n : L^{n-1}\tau_2 \vdash \mathsf{ret}\, h_n :: t_n : \mathbb{M}\,0\,L^n\tau_2}$$

D1.1:

$$\frac{\dfrac{}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2 \vdash f[][]\ !g_u\ t : \mathbb{M}\,0\,L^{n-1}\tau_2} \qquad D1.2}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2, t : L^{n-1}([c]\,\tau_1) \vdash E_{0.4} : \mathbb{M}\,0\,L^n\tau_2}$$

D1.0:

$$\frac{\dfrac{}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_e : \tau_1 \vdash (g_u\ h_e) : \mathbb{M}\,c\,\tau_2} \qquad D1.1}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_e : \tau_1, t : L^{n-1}([c]\,\tau_1) \vdash E_{0.3} : \mathbb{M}\,c\,L^n\tau_2}$$

D1:

$$\frac{\dfrac{}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h : [c]\,\tau_1 \vdash h : [c]\,\tau_1} \qquad D1.0}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h : [c]\,\tau_1, t : L^{n-1}([c]\,\tau_1) \vdash E_{0.2} : \mathbb{M}\,0\,L^n\tau_2}$$

D0:

$$\frac{\dfrac{\dfrac{}{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; . \vdash nil : L^0\tau_2}}{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; . \vdash \mathsf{ret}\, nil : \mathbb{M}\,0\,L^n\tau_2}}{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; . \vdash E_{0.1} : \mathbb{M}\,0\,L^n\tau_2}$$

Main derivation:

$$\frac{\dfrac{\dfrac{\dfrac{}{.; n, c; f : T_0; g :!T_{1.1} \vdash g :!T_{1.1}} \qquad \dfrac{\dfrac{}{.; n, c; .; f : T_0, g_u : T_{1.1}; l : T_{1.2} \vdash l : T_{1.2}} \qquad D0 \qquad D1}{.; n, c; .; f : T_0, g_u : T_{1.1}; l : T_{1.2} \vdash E_0 : \mathbb{M}\,0\,L^n\tau_2}}{.; n, c; .; f : T_0; g :!T_{1.1}, l : T_{1.2} \vdash E_3 : \mathbb{M}\,0\,L^n\tau_2}}{.; n, c; .; f : T_0; . \vdash E_2 : T_1}}{\dfrac{.; .; .; f : T_0; . \vdash E_1 : T_0}{.; .; .; .; . \vdash E : T_0}}$$

3.6 Okasaki's implicit queue

Typing rules for value constructors and case analysis

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C0} : \mathit{Queue}\ \tau}\ \text{T-C0} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C1}\ e : \mathit{Queue}\ \tau}\ \text{T-C1}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\tau \otimes \mathit{Queue}\ (\tau \otimes \tau))}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C2}\ e : \mathit{Queue}\ \tau}\ \text{T-C2}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [0]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\tau \otimes \mathit{Queue}\ (\tau \otimes \tau)) \otimes \tau)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C3}\ e : \mathit{Queue}\ \tau}\ \text{T-C3}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [2]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\tau \otimes \tau) \otimes \mathit{Queue}\ (\tau \otimes \tau))}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C4}\ e : \mathit{Queue}\ \tau}\ \text{T-C4}$$

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(((\tau \otimes \tau) \otimes \mathit{Queue}\ (\tau \otimes \tau)) \otimes \tau)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \mathit{C5}\ e : \mathit{Queue}\ \tau}\ \text{T-C5}$$

$$\frac{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\mathit{Queue}\ \tau) \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_0 : \tau' \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau \vdash e_1 : \tau' \qquad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\tau \otimes \mathit{Queue}\ (\tau \otimes \tau)) \vdash e_2 : \tau' \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [0]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\tau \otimes \mathit{Queue}\ (\tau \otimes \tau)) \otimes \tau) \vdash e_3 : \tau' \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [2]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\tau \otimes \tau) \otimes \mathit{Queue}\ (\tau \otimes \tau)) \vdash e_4 : \tau' \\ \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(((\tau \otimes \tau) \otimes \mathit{Queue}\ (\tau \otimes \tau)) \otimes \tau) \vdash e_5 : \tau'\end{array}}{\begin{array}{c}\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \\ \mathsf{case}\ e\ \mathsf{of}\ |\mathit{C0} \mapsto e_0\ |\mathit{C1}\ x \mapsto e_1\ |\mathit{C2}\ x \mapsto e_2\ |\mathit{C3}x \mapsto e_3\ |\mathit{C4}x \mapsto e_4\ |\mathit{C5}x \mapsto e_5 : \tau'\end{array}}\ \text{T-caseIQ}$$

$\mathit{snoc} : [2]\,\mathbf{1} \multimap \forall\alpha.\mathit{Queue}\ \ \alpha \multimap \alpha \multimap \mathbb{M}\,0\,\mathit{Queue}\ \ \alpha$
$\mathsf{fix}\ \ \mathit{snoc}.\lambda p.\Lambda.\lambda\ \ q\ \ a.$
$- = \mathsf{release}\,p\ \mathsf{in}\ \ - = \uparrow^1;\ \mathsf{ret}$
$\quad\mathsf{case}\ \ q\ \ \mathsf{of}$
$\quad\ \ |\mathit{C0} \mapsto \mathsf{ret}\ \mathit{C1}\ \ a$

$\quad\ \ |\mathit{C1}\ \ x \mapsto \mathsf{ret}\ \mathit{C4}\ \ (\lambda p''.\ \mathsf{ret}\langle\!\langle\langle\!\langle x, a\rangle\!\rangle, \mathit{C0}\rangle\!\rangle)$

$\quad\ \ |\mathit{C2}\ \ x \mapsto$
$\quad\quad \mathsf{bind}\,p' = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\ \ \mathsf{bind}\,x' = x\ \ p'\ \mathsf{in}$
$\quad\quad\quad \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = x'\ \mathsf{in}$
$\quad\quad\quad\ \ \mathsf{ret}(\mathit{C3}\ \ (\lambda p''.\langle\!\langle\langle\!\langle f, m\rangle\!\rangle, a\rangle\!\rangle))$

$\quad\ \ |\mathit{C3}\ \ x \mapsto$
$\quad\quad \mathsf{bind}\,p' = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\ \ \mathsf{bind}\,x' = x\ \ p'\ \mathsf{in}\ \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x'\ \mathsf{in}$
$\quad\quad\quad \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = fm\ \mathsf{in}\ \ \mathsf{bind}\,p_o = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\quad\ \ \mathsf{ret}\ \mathit{C2}\ \ (\lambda p''.$
$\quad\quad\quad\quad\quad\quad - = \mathsf{release}\,p_o\ \mathsf{in}\ - = \mathsf{release}\,p''\ \mathsf{in}\ \mathsf{bind}\,p''' = \mathsf{store}()\ \mathsf{in}$
$\quad\quad\quad\quad\quad\quad \mathsf{bind}\,m' = \mathit{snoc}\ \ p'''\ \ m\ \ (r, a))\ \mathsf{in}\ \mathsf{ret}\langle\!\langle f, m'\rangle\!\rangle$

$|C4\ \ x \mapsto$
$\mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}$
$\mathsf{ret}\ C5\ \ (\lambda p''.$
$- = \mathsf{release}\, p'\ \mathsf{in}\ - = \mathsf{release}\, p''\ \mathsf{in}$
$\mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = x\ \ p'''\ \mathsf{in}$
$\mathsf{ret} \langle\!\langle \langle\!\langle f, m \rangle\!\rangle, a \rangle\!\rangle$

$|C5\ \ x \mapsto$
$\mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}$
$\mathsf{bind}\, x' = x\ \ p'\ \mathsf{in}$
$\mathsf{let} \langle\!\langle fm, r \rangle\!\rangle = x'\ \mathsf{in}\ \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = fm\ \mathsf{in}$
$\mathsf{ret}(C4\ \ (\lambda p''.$
$\mathsf{bind}\, m' = snoc\ \ p''\ \ m\ \mathsf{in}\ \mathsf{ret} \langle\!\langle f, m' \rangle\!\rangle)$

Listing 2: snoc function

$E_{0.0} = - = \mathsf{release}\, p\ \mathsf{in} E_{0.1}$
$E_{0.1} = - = \uparrow^1; E_{0.2}$
$E_{0.2} = \mathsf{case}\ q\ \mathsf{of} |C0 \mapsto E_0 |C1\ \ x \mapsto E_1 |C2\ \ x \mapsto E_2 |C3\ \ x \mapsto E_3 |C4\ \ x \mapsto E_4 |C5\ \ x \mapsto E_5$
$E_0 = \mathsf{ret}(C1\ \ a)$
$E_1 = \mathsf{ret}\ C4\ \ (\lambda p''. \mathsf{ret} \langle\!\langle \langle\!\langle x, a \rangle\!\rangle, C0 \rangle\!\rangle)$
$E_2 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{2.1}$
$E_{2.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{2.2}$
$E_{2.2} = \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = x'\ \mathsf{in}\ E_{2.3}$
$E_{2.3} = \mathsf{ret}(C3\ \ (\lambda p''. \langle\!\langle \langle\!\langle f, m \rangle\!\rangle, a \rangle\!\rangle))$
$E_3 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{3.1}$
$E_{3.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{3.2}$
$E_{3.2} = \mathsf{let} \langle\!\langle fm, r \rangle\!\rangle = x'\ \mathsf{in}\ E_{3.3}$
$E_{3.3} = \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = fm\ \mathsf{in}\ E_{3.31}$
$E_{3.31} = \mathsf{bind}\, p_o = \mathsf{store}()\ \mathsf{in}\ E_{3.4}$
$E_{3.4} = \mathsf{ret}\ C2\ \ (\lambda p''. E_{3.41})$
$E_{3.41} = - = \mathsf{release}\, p_o\ \mathsf{in}\ - = \mathsf{release}\, p''\ \mathsf{in}\ \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.42}$
$E_{3.42} = \mathsf{bind}\, m' = snoc\ p'''\ m\ (r, a)\ \mathsf{in}\ \mathsf{ret} \langle\!\langle f, m' \rangle\!\rangle$
$E_4 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{4.1}$
$E_{4.1} = \mathsf{ret}\ C5\ \ (\lambda p''. E_{4.11})$
$E_{4.11} = - = \mathsf{release}\, p'\ \mathsf{in}\ - = \mathsf{release}\, p''\ \mathsf{in}\ E_{4.12}$
$E_{4.12} = \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = x\ p'''\ \mathsf{in}\ E_{4.13}$
$E_{4.13} = \mathsf{ret} \langle\!\langle \langle\!\langle f, m \rangle\!\rangle, a \rangle\!\rangle$
$E_5 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{5.1}$
$E_{5.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{5.2}$
$E_{5.2} = \mathsf{let} \langle\!\langle fm, r \rangle\!\rangle = x'\ \mathsf{in}\ E_{5.3}$
$E_{5.3} = \mathsf{let} \langle\!\langle f, m \rangle\!\rangle = fm\ \mathsf{in}\ E_{5.4}$
$E_{5.4} = \mathsf{ret}(C4\ \ (\lambda p''.\ \mathsf{bind}\, m' = snoc\ p''\ m\ \mathsf{in}\ \mathsf{ret} \langle\!\langle f, m' \rangle\!\rangle))$

$T_{0.0} = [2]\, \mathbf{1} \multimap \forall \alpha. Queue\ \ \alpha \multimap \alpha \multimap \mathbb{M}\, 0\ Queue\ \ \alpha$
$T_0 = \mathbb{M}\, 0\ Queue\ \ \alpha$

$T_1 = \mathbb{M}\,1\; Queue\ \alpha$
$T_2 = \mathbb{M}\,2\; Queue\ \alpha$
$T_3 = \mathbb{M}\,0\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{3.1} = (\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{3.2} = Queue\ (\alpha \otimes \alpha)$
$T_4 = \mathbb{M}\,0\,(\alpha \otimes Queue\ (\alpha \otimes \alpha) \otimes \alpha)$
$T_{4.1} = (\alpha \otimes Queue\ (\alpha \otimes \alpha) \otimes \alpha)$
$T_{4.2} = \alpha \otimes Queue\ (\alpha \otimes \alpha)$
$T_{4.3} = Queue\ (\alpha \otimes \alpha)$
$T_5 = [2]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)$
$T_{5.1} = \mathbb{M}\,0\,(\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)$
$T_{5.2} = (\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)$
$T_{5.3} = (\alpha \otimes \alpha)$
$T_{5.4} = Queue\ (\alpha \otimes \alpha)$
$T_6 = [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha) \otimes \alpha)$
$T_{6.1} = \mathbb{M}\,0\,((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha) \otimes \alpha)$
$T_{6.2} = ((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha) \otimes \alpha)$
$T_{6.3} = (\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)$
$T_{6.4} = (\alpha \otimes \alpha)$
$T_{6.5} = Queue\ (\alpha \otimes \alpha)$
$T_7 = \mathbb{M}\,0\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{7.1} = \mathbb{M}\,1\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{7.2} = \mathbb{M}\,2\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_8 = \mathbb{M}\,0\,(((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{8.1} = ((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha)) \otimes \alpha$
$T_9 = \mathbb{M}\,0\,((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha))$
$T_{9.1} = ((\alpha \otimes \alpha) \otimes Queue\ (\alpha \otimes \alpha))$

D5.5:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2]\,\mathbf{1}, m' : Queue\ (\alpha \otimes \alpha) \vdash \langle\langle f, m' \rangle\rangle : T_{9.1}}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2]\,\mathbf{1}, m' : Queue\ (\alpha \otimes \alpha) \vdash \mathsf{ret}\langle\langle f, m' \rangle\rangle : T_9}$$

D5.4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, m : T_{6.5}, p'' : [2]\,\mathbf{1} \vdash S\ p''\ []\ m\ \langle\langle r, a \rangle\rangle : \mathbb{M}\,0\,(Queue\ (\alpha \otimes \alpha))} \\ D5.5\end{array}}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2]\,\mathbf{1} \vdash \mathsf{bind}\, m' = S\ p''\ []\ m\ \langle\langle r, a \rangle\rangle\ \mathsf{in}\ \mathsf{ret}\langle\langle f, m' \rangle\rangle : T_9}}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash (\lambda p''.\,\mathsf{bind}\, m' = S\ p''\ []\ m\ \langle\langle r, a \rangle\rangle\ \mathsf{in}\ \mathsf{ret}\langle\langle f, m' \rangle\rangle) : [2]\,\mathbf{1} \multimap T_9}}{\begin{array}{c}\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash \\ (C4\ (\lambda p''.\,\mathsf{bind}\, m' = S\ p''\ []\ m\ \langle\langle r, a \rangle\rangle\ \mathsf{in}\ \mathsf{ret}\langle\langle f, m' \rangle\rangle)) : Queue\ \alpha\end{array}}}{\dfrac{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash \mathsf{ret}(C4\ (\lambda p''.\,\mathsf{bind}\, m' = S\ p''\ []\ m\ \langle\langle r, a \rangle\rangle\ \mathsf{in}\ \mathsf{ret}\langle\langle f, m' \rangle\rangle)) : T_0}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash E_{5.4} : T_0}}$$

D5.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; fm : T_{6.3} \vdash fm : T_{6.3}} \qquad D5.4}{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{6.3}, r : \alpha \vdash \mathsf{let}\langle\langle f, m \rangle\rangle = fm\ \mathsf{in}\ E_{5.4} : T_0}}{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{6.3}, r : \alpha \vdash E_{5.3} : T_0}$$

D5.2:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x':T_{6.2}\vdash x':T_{6.2}}\qquad D5.3}{\alpha;.;.;S:T_{0.0};a:\alpha,x':T_{6.2}\vdash \mathsf{let}\langle\langle fm,r\rangle\rangle = x' \ \mathsf{in}\ E_{5.3}:T_0}}{\alpha;.;.;S:T_{0.0};a:\alpha,x':T_{6.2}\vdash E_{5.2}:T_0}$$

D5.1:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x:T_6,p':[1]\,\mathbf{1}\vdash x\ p':T_{6.1}}\qquad D5.2}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_6,p':[1]\,\mathbf{1}\vdash \mathsf{bind}\,x' = x\ p'\ \mathsf{in}\ E_{5.2}:T_0}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_6,p':[1]\,\mathbf{1}\vdash E_{5.1}:T_0}$$

D5:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};.\vdash \mathsf{store}():\mathbb{M}\,1\,([1]\,\mathbf{1})}\qquad D5.1}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_6\vdash E_5:T_1}$$

D4.5:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,f:T_{5.3},m:T_{5.4}\vdash \langle\langle\langle\langle f,m\rangle\rangle,a\rangle\rangle:T_{8.1}}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,f:T_{5.3},m:T_{5.4}\vdash \mathsf{ret}\langle\langle\langle\langle f,m\rangle\rangle,a\rangle\rangle:T_8}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,f:T_{5.3},m:T_{5.4}\vdash E_{4.13}:T_8}$$

D4.4:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x:T_5,p''':[2]\,\mathbf{1}\vdash x\ p''':T_{5.1}}\qquad D4.5}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,p''':[2]\,\mathbf{1}\vdash \mathsf{let}\langle\langle f,m\rangle\rangle = x\ p'''\ \mathsf{in}\ E_{4.13}:T_8}$$

D4.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5\vdash \mathsf{store}():\mathbb{M}\,2\,([2]\,\mathbf{1})}\qquad D4.4}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5\vdash \mathsf{bind}\,p''' = \mathsf{store}()\ \mathsf{in}\ \mathsf{let}\langle\langle f,m\rangle\rangle = x\ p'''\ \mathsf{in}\ E_{4.13}:T_{8.2}}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5\vdash E_{4.12}:T_{8.2}}$$

D4.2:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};p'':[1]\,\mathbf{1}\vdash p'':[1]\,\mathbf{1}}\qquad D4.3}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,p'':\vdash - = \mathsf{release}\,p''\ \mathsf{in}\ E_{4.12}:T_{8.1}}$$

D4.11:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};p':[1]\,\mathbf{1}\vdash p':[1]\,\mathbf{1}}\qquad D4.2}{\alpha;.;.;S:T_{0.0};a:\alpha,x:T_5,p':[1]\,\mathbf{1},p'':[1]\,\mathbf{1}\vdash - = \mathsf{release}\,p'\ \mathsf{in}\ - = \mathsf{release}\,p''\ \mathsf{in}\ E_{4.12}:T_8}$$

D4.1:

$$\dfrac{\dfrac{\dfrac{\dfrac{D4.11}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[2]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\alpha\otimes\alpha)\otimes \mathit{Queue}\ (\alpha\otimes\alpha),p':[1]\,\mathbf{1},p'':[1]\,\mathbf{1}\vdash E_{4.11}:T_8}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[2]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\alpha\otimes\alpha)\otimes \mathit{Queue}\ (\alpha\otimes\alpha),p':[1]\,\mathbf{1}\vdash (\lambda p''.E_{4.11}):[1]\,\mathbf{1}\multimap T_8}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[2]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\alpha\otimes\alpha)\otimes \mathit{Queue}\ (\alpha\otimes\alpha),p':[1]\,\mathbf{1}\vdash \mathit{C5}\ (\lambda p''.E_{4.11}):\mathit{Queue}\ \alpha}}{\dfrac{\alpha;.;.;S:T_{0.0};a:\alpha,x:[2]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\alpha\otimes\alpha)\otimes \mathit{Queue}\ (\alpha\otimes\alpha),p':[1]\,\mathbf{1}\vdash \mathsf{ret}\ \mathit{C5}\ (\lambda p''.E_{4.11}):T_0}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[2]\,\mathbf{1}\multimap\mathbb{M}\,0\,(\alpha\otimes\alpha)\otimes \mathit{Queue}\ (\alpha\otimes\alpha),p':[1]\,\mathbf{1}\vdash E_{4.1}:T_0}}$$

D4:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,1\,([1]\,\mathbf{1})}\ D4.1}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha) \vdash E_4 : T_1}$$

D3.43:

$$\dfrac{}{\alpha; .; .; S : T_{0.0}; f : \alpha, m' : \mathit{Queue}\ (\alpha \otimes \alpha) \vdash \mathsf{ret}\langle\langle f, m' \rangle\rangle : T_7}$$

D3.42:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2]\,\mathbf{1} \vdash S\ p'''\ []\ m\ (r, a) : \mathbb{M}\,0\,(\mathit{Queue}\ (\alpha \otimes \alpha))} \quad D3.43}{\begin{array}{c}\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2]\,\mathbf{1} \vdash \\ \mathsf{bind}\, m' = S\ p'''\ []\ m\ (r, a)\ \mathsf{in}\ \mathsf{ret}\langle\langle f, m' \rangle\rangle : T_7\end{array}}}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2]\,\mathbf{1} \vdash E_{3.42} : T_7}$$

D3.41:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,2\,([2]\,\mathbf{1})} \quad D3.42}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha \vdash \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.42} : T_{7.2}}$$

D3.401:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; p'' : [1]\,\mathbf{1} \vdash p'' : [1]\,\mathbf{1}} \quad D3.41}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p'' : [1]\,\mathbf{1} \vdash - = \mathsf{release}\, p''\ \mathsf{in}\ \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.42} : T_{7.1}}$$

D3.40:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; p_o : [1]\,\mathbf{1} \vdash p_o : [1]\,\mathbf{1}} \quad D3.401}{\begin{array}{c}\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1}, p'' : [1]\,\mathbf{1} \vdash \\ - = \mathsf{release}\, p_o\ \mathsf{in}\ - = \mathsf{release}\, p''\ \mathsf{in}\ \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.42} : T_7\end{array}}}{\begin{array}{c}\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1} \vdash \\ \lambda p''.- = \mathsf{release}\, p_o\ \mathsf{in}\ - = \mathsf{release}\, p''\ \mathsf{in}\ \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.42} : [1]\,\mathbf{1} \multimap T_7\end{array}}}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1} \vdash (\lambda p''.E_{3.41}) : [1]\,\mathbf{1} \multimap T_7}$$

D3.4:

$$\dfrac{\dfrac{\dfrac{D3.40}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1} \vdash \mathit{C2}\ (\lambda p''.E_{3.41}) : \mathit{Queue}\ \alpha}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1} \vdash \mathsf{ret}\ \mathit{C2}\ (\lambda p''.E_{3.41}) : T_1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1]\,\mathbf{1} \vdash E_{3.4} : T_1}$$

D3.31:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash \mathsf{store}() : \mathbb{M}\,1\,[1]\,\mathbf{1}} \quad D3.4}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash \mathsf{bind}\, p_o = \mathsf{store}()\ \mathsf{in}\ E_{3.4} : T_1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash E_{3.31} : T_1}$$

D3.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};fm:T_{4.2}\vdash fm:T_{4.2}}\qquad D3.4}{\alpha;.;.;S:T_{0.0};a:\alpha,fm:T_{4.2},r:\alpha\vdash \mathsf{let}\langle\!\langle f,m\rangle\!\rangle = fm\ \mathsf{in}\ E_{3.31}:T_1}}{\alpha;.;.;S:T_{0.0};a:\alpha,fm:T_{4.2},r:\alpha\vdash E_{3.3}:T_1}$$

D3.2:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x':T_{4.1}\vdash x':T_{4.1}}\qquad D3.3}{\alpha;.;.;S:T_{0.0};.\vdash \mathsf{let}\langle\!\langle fm,r\rangle\!\rangle = x'\ \mathsf{in}\ E_{3.3}:T_1}}{\alpha;.;.;S:T_{0.0};a:\alpha,x':T_{4.1}\vdash E_{3.2}:T_1}$$

D3.1:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x:[0]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha)\otimes\alpha),p':[0]\,\mathbf{1}\vdash x\ p':T_4}\qquad D3.2}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[0]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha)\otimes\alpha)\vdash E_{3.1}:T_1}$$

D3:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};\vdash \mathsf{store}():\mathbb{M}\,0\,([0]\,\mathbf{1})}\qquad D3.1}{\alpha;.;.;S:T_{0.0};a:\alpha,x:[0]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha)\otimes\alpha)\vdash E_3:T_1}$$

D2.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};q:\mathit{Queue}\ \alpha,a:\alpha,f:\alpha,m:T_{3.2}\vdash (\mathit{C3}\ (\lambda p''.\langle\!\langle\langle\!\langle f,m\rangle\!\rangle,a\rangle\!\rangle)):\mathit{Queue}\ \alpha}}{\alpha;.;.;S:T_{0.0};q:\mathit{Queue}\ \alpha,a:\alpha,f:\alpha,m:T_{3.2}\vdash \mathsf{ret}(\mathit{C3}\ (\lambda p''.\langle\!\langle\langle\!\langle f,m\rangle\!\rangle,a\rangle\!\rangle)):T_0}}{\alpha;.;.;S:T_{0.0};q:\mathit{Queue}\ \alpha,a:\alpha,f:\alpha,m:T_{3.2}\vdash E_{2.3}:T_0}$$

D2.2:

$$\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x':T_{3.1}\vdash x':T_{3.1}}\qquad D2.3}{\alpha;.;.;S:T_{0.0};a:\alpha,x':T_{3.1}\vdash \mathsf{let}\langle\!\langle f,m\rangle\!\rangle = x'\ \mathsf{in}\ E_{2.3}:T_0}}{\alpha;.;.;S:T_{0.0};a:\alpha,x':T_{3.1}\vdash E_{2.2}:T_0}$$

D2.1:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};x:([1]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha))),p':[1]\,\mathbf{1}\vdash x\ p':T_3}\qquad D2.2}{\alpha;.;.;S:T_{0.0};a:\alpha,x:([1]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha))),p':[1]\,\mathbf{1}\vdash E_{2.1}:T_0}$$

D2:

$$\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};.\vdash \mathsf{store}():\mathbb{M}\,1\,([1]\,\mathbf{1})}\qquad D2.1}{\alpha;.;.;S:T_{0.0};a:\alpha,x:([1]\,\mathbf{1}\multimap \mathbb{M}\,0\,(\alpha\otimes \mathit{Queue}\ (\alpha\otimes\alpha)))\vdash E_2:T_1}$$

D1:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha;.;.;S:T_{0.0};a:\alpha,x:\alpha\vdash \mathit{C4}\ (\lambda p''.\,\mathsf{ret}\langle\!\langle\langle\!\langle x,a\rangle\!\rangle,\mathit{C0}\rangle\!\rangle):\mathit{Queue}\ \alpha}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:\alpha\vdash \mathsf{ret}\ \mathit{C4}\ (\lambda p''.\,\mathsf{ret}\langle\!\langle\langle\!\langle x,a\rangle\!\rangle,\mathit{C0}\rangle\!\rangle):T_0}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:\alpha\vdash \mathsf{ret}\ \mathit{C4}\ (\lambda p''.\,\mathsf{ret}\langle\!\langle\langle\!\langle x,a\rangle\!\rangle,\mathit{C0}\rangle\!\rangle):T_1}}{\alpha;.;.;S:T_{0.0};a:\alpha,x:\alpha\vdash E_1:T_1}$$

D0:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash \mathit{C1}\ a : \mathit{Queue}\ \alpha}}{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash \mathsf{ret}(\mathit{C1}\ a) : \mathbb{M}\, 1\, \mathit{Queue}\ \alpha}}{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash E_0 : T_1}$$

D0.2:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; q : \mathit{Queue}\ \alpha \vdash q : \mathit{Queue}\ \alpha} \quad D0 \quad D1 \quad D2 \quad D3 \quad D4 \quad D5}{\alpha; .; .; S : T_{0.0}; q : \mathit{Queue}\ \alpha, a : \alpha \vdash E_{0.2} : T_1}$$

D0.1:

$$\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; . \vdash \uparrow^1 : \mathbb{M}\, 1\, \mathbf{1}} \quad D0.2}{\alpha; .; .; S : T_{0.0}; q : \mathit{Queue}\ \alpha, a : \alpha \vdash E_{0.1} : T_2}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; S : T_{0.0}; p : [2]\, \mathbf{1} \vdash p : [2]\, \mathbf{1}} \quad D0.1}{\alpha; .; .; S : T_{0.0}; p : [2]\, \mathbf{1}, q : \mathit{Queue}\ \alpha, a : \alpha \vdash E_{0.0} : T_0}}{.; .; .; .; . \vdash \mathsf{fix} f.\lambda p.\Lambda.\lambda q.\lambda a.E_{0.0} : T_{0.0}}$$

$\mathit{head} : [3]\, \mathbf{1} \multimap \forall \alpha.\mathit{Queue}\ \ \alpha \multimap \mathbb{M}\, 0\, \alpha$
$\mathit{head} \triangleq \lambda p.\Lambda.\lambda\ \ q.$
$\quad \mathsf{bind}\, ht = \mathit{headTail}\ \ p\ \ []\ \ q\ \mathsf{in}\ \mathsf{ret}\, \mathsf{fst}(ht)$

Listing 3: head function

$E_0 = \mathsf{bind}\, ht = \mathit{headTail}\ p\ []\ q\ \mathsf{in}\ E_1$
$E_1 = \mathsf{ret}(\mathsf{fst}(ht))$

$T_0 = [3]\, \mathbf{1} \multimap \forall \alpha.\mathit{Queue}\ \alpha \multimap \mathbb{M}\, 0\, \alpha$

D0:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; .; q : \mathit{Queue}\ \alpha, ht : (\alpha \otimes \mathit{Queue}\ \alpha) \vdash \mathsf{fst}(ht) : \alpha}}{\alpha; .; .; .; q : \mathit{Queue}\ \alpha, ht : (\alpha \otimes \mathit{Queue}\ \alpha) \vdash \mathsf{ret}(\mathsf{fst}(ht)) : \mathbb{M}\, 0\, \alpha}}{\alpha; .; .; .; q : \mathit{Queue}\ \alpha, ht : (\alpha \otimes \mathit{Queue}\ \alpha) \vdash E_1 : \mathbb{M}\, 0\, \alpha}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; .; q : \mathit{Queue}\ \alpha \vdash \mathit{headTail}\ p\ []\ q : \mathbb{M}\, 0\, (\alpha \otimes \mathit{Queue}\ \alpha)} \quad D0}{\alpha; .; .; .; q : \mathit{Queue}\ \alpha \vdash \mathsf{bind}\, ht = \mathit{headTail}\ p\ []\ q\ \mathsf{in}\ E_1 : \mathbb{M}\, 0\, \alpha}}{\alpha; .; .; .; p : [3]\, \mathbf{1}, q : \mathit{Queue}\ \alpha \vdash E_0 : \mathbb{M}\, 0\, \alpha}}{.; .; .; .; . \vdash \lambda p.\Lambda.\lambda q.E_0 : T_0}$$

$$
\begin{array}{l}
tail : [3]\,\mathbf{1} \multimap \forall \alpha. Queue \;\; \alpha \multimap \mathbb{M}\,0\,(Queue \;\; \alpha) \\
tail \triangleq \lambda p.\Lambda.\lambda \;\; q. \\
\;\; \mathsf{bind}\, ht = headTail \;\; p \;\; [] \;\; q \;\mathsf{in}\; \mathsf{ret}\,\mathsf{snd}(ht)
\end{array}
$$

Listing 4: tail function

$$
\begin{array}{l}
E_0 = \mathsf{bind}\, ht = headTail \; p \; [] \; q \;\mathsf{in}\; E_1 \\
E_1 = \mathsf{ret}(\mathsf{snd}(ht))
\end{array}
$$

$$T_0 = [3]\,\mathbf{1} \multimap \forall \alpha. Queue \; \alpha \multimap \mathbb{M}\,0\,(Queue \; \alpha)$$

D0:

$$
\dfrac{\dfrac{\dfrac{}{\alpha; .; .; .; q : Queue \; \alpha, ht : (\alpha \otimes Queue \; \alpha) \vdash \mathsf{snd}(ht) : Queue \; \alpha}}{\alpha; .; .; .; q : Queue \; \alpha, ht : (\alpha \otimes Queue \; \alpha) \vdash \mathsf{ret}(\mathsf{snd}(ht)) : \mathbb{M}\,0\,(Queue \; \alpha)}}{\alpha; .; .; .; q : Queue \; \alpha, ht : (\alpha \otimes Queue \; \alpha) \vdash E_1 : \mathbb{M}\,0\,(Queue \; \alpha)}
$$

Main derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; .; q : Queue \; \alpha \vdash headTail \; p \; [] \; q : \mathbb{M}\,0\,(\alpha \otimes Queue \; \alpha)} \quad D0}{\alpha; .; .; .; q : Queue \; \alpha \vdash \mathsf{bind}\, ht = headTail \; p \; [] \; q \;\mathsf{in}\; E_1 : \mathbb{M}\,0\,(Queue \; \alpha)}}{\alpha; .; .; .; p : [3]\,\mathbf{1}, q : Queue \; \alpha \vdash E_0 : \mathbb{M}\,0\,(Queue \; \alpha)}}{.; .; .; .; . \vdash \lambda p.\Lambda.\lambda q.E_0 : T_0}
$$

$$
\begin{array}{l}
headTail : [3]\,\mathbf{1} \multimap \forall \alpha. Queue \;\; \alpha \multimap \mathbb{M}\,0\,(\alpha \otimes Queue \;\; \alpha) \\
headTail \triangleq \mathsf{fix} \;\; HT.\lambda p.\Lambda.\lambda \;\; q. \\
- = \mathsf{release}\, p \;\mathsf{in}\;\; - = \uparrow^1; \;\mathsf{ret} \\
\;\; \mathsf{case} \;\; q \;\; \mathsf{of} \\
\;\;\;\; |\mathit{C0} \mapsto \mathsf{fix} x.x \\
\\
\;\;\;\; |\mathit{C1} \;\; x \mapsto \mathsf{ret}\langle\!\langle x, \mathit{C0} \rangle\!\rangle \\
\\
\;\;\;\; |\mathit{C2} \;\; x \mapsto \\
\;\;\;\;\;\; \mathsf{bind}\, p' = \mathsf{store}() \;\mathsf{in}\;\; \mathsf{bind}\, p_o = \mathsf{store}() \;\mathsf{in} \\
\;\;\;\;\;\;\;\; \mathsf{bind}\, x' = x \;\; p' \;\mathsf{in}\; \mathsf{let}\langle\!\langle f, m \rangle\!\rangle = x' \;\mathsf{in} \\
\;\;\;\;\;\;\;\;\;\; \mathsf{ret}\langle\!\langle f, (\mathit{C4} \;\; (\lambda p''. - = \mathsf{release}\, p_o \;\mathsf{in}\; - = \mathsf{release}\, p'' \;\mathsf{in}\; \mathsf{bind}\, p_r = \mathsf{store}() \;\mathsf{in}\; HT \;\; p_r \;\; [] \;\; m)) \rangle\!\rangle \\
\\
\;\;\;\; |\mathit{C3} \;\; x \mapsto \\
\;\;\;\;\;\; \mathsf{bind}\, p' = \mathsf{store}() \;\mathsf{in}\;\; \mathsf{bind}\, p_o = \mathsf{store}() \;\mathsf{in} \\
\;\;\;\;\;\;\;\; \mathsf{bind}\, x' = x \;\; p' \;\mathsf{in}\;\; \mathsf{let}\langle\!\langle fm, r \rangle\!\rangle = x' \;\mathsf{in}\;\; \mathsf{let}\langle\!\langle f, m \rangle\!\rangle = fm \;\mathsf{in} \\
\;\;\;\;\;\;\;\; \mathsf{ret}\langle\!\langle f, (\mathit{C5} \;\; (\lambda p''. - = \mathsf{release}\, p_o \;\mathsf{in}\; - = \mathsf{release}\, p'' \;\mathsf{in} \\
\qquad\qquad\qquad\qquad \mathsf{bind}\, p''' = \mathsf{store}() \;\mathsf{in}\;\; \mathsf{bind}\, ht = HT \;\; p''' \;\; [] \;\; m \;\mathsf{in}\; \mathsf{ret}\langle\!\langle ht, r \rangle\!\rangle)) \rangle\!\rangle \\
\\
\;\;\;\; |\mathit{C4} \;\; x \mapsto \\
\;\;\;\;\;\; \mathsf{bind}\, p' = \mathsf{store}() \;\mathsf{in}\; \mathsf{bind}\, x' = x \;\; p' \;\mathsf{in}\; \mathsf{let}\langle\!\langle f, m \rangle\!\rangle = x' \;\mathsf{in}\; \mathsf{let}\langle\!\langle f_1, f_2 \rangle\!\rangle = f \;\mathsf{in} \\
\;\;\;\;\;\; \mathsf{ret}\langle\!\langle f_1, \mathit{C2} \;\; (\lambda p''.\, \mathsf{ret}\langle\!\langle f_2, m \rangle\!\rangle) \rangle\!\rangle
\end{array}
$$

$|C5\ \ x \mapsto$

$\quad \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ \mathsf{bind}\, x' = x\ \ p'\ \mathsf{in}\ \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x'\ \mathsf{in}\ \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = fm\ \mathsf{in}\ \mathsf{let}\langle\!\langle f_1, f_2\rangle\!\rangle = f\ \mathsf{in}$

$\qquad \mathsf{ret}\langle\!\langle f_1, (C3\ \ (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle))\rangle\!\rangle$

Listing 5: head and tail function

$E_{0.0} = \mathsf{fix} HT.\lambda p.\Lambda.\lambda\ q.E_{0.1}$
$E_{0.1} = - = \mathsf{release}\, p\ \mathsf{in} - = \uparrow^1; E_{0.2}$
$E_{0.2} = \mathsf{case}\ q\ \mathsf{of}|C0 \mapsto E_0|C1\ \ x \mapsto E_1|C2\ \ x \mapsto E_2|C3\ \ x \mapsto E_3|C4\ \ x \mapsto E_4|C5\ \ x \mapsto E_5$
$E_0 = \mathsf{fix} x.x$
$E_1 = \mathsf{ret}\langle\!\langle x, C0\rangle\!\rangle$
$E_2 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{2.0}$
$E_{2.0} = \mathsf{bind}\, p_o = \mathsf{store}()\ \mathsf{in}\ E_{2.1}$
$E_{2.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{2.11}$
$E_{2.11} = \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = x'\ \mathsf{in}\ E_{2.2}$
$E_{2.2} = \mathsf{ret}\langle\!\langle f, (C4\ \ (\lambda p''.E_{2.3}))\rangle\!\rangle$
$E_{2.3} = - = \mathsf{release}\, p_o\ \mathsf{in}\ E_{2.4}$
$E_{2.4} = - = \mathsf{release}\, p''\ \mathsf{in}\ E_{2.5}$
$E_{2.5} = \mathsf{bind}\, p_r = \mathsf{store}()\ \mathsf{in}\ HT\ \ p_r\ \ []\ \ m$
$E_3 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{3.0}$
$E_{3.0} = \mathsf{bind}\, p_o = \mathsf{store}()\ \mathsf{in}\ E_{3.1}$
$E_{3.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{3.11}$
$E_{3.11} = \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x'\ \mathsf{in}\ E_{3.12}$
$E_{3.12} = \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = fm\ \mathsf{in}\ E_{3.2}$
$E_{3.2} = \mathsf{ret}\langle\!\langle f, E_{3.3}\rangle\!\rangle$
$E_{3.3} = C5\ \ (\lambda p''.E_{3.31})$
$E_{3.4} = - = \mathsf{release}\, p_o\ \mathsf{in}\ E_{3.41}$
$E_{3.41} = \mathsf{release}\, p''\ \mathsf{in}\ E_{3.5}$
$E_{3.5} = \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.6}$
$E_{3.6} = \mathsf{bind}\, ht = HT\ \ p'''\ \ []\ \ m\ \mathsf{in}\ \mathsf{ret}\langle\!\langle ht, r\rangle\!\rangle$
$E_4 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{4.1}$
$E_{4.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{4.2}$
$E_{4.2} = \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = x'\ \mathsf{in}\ E_{4.3}$
$E_{4.3} = \mathsf{let}\langle\!\langle f_1, f_2\rangle\!\rangle = f\ \mathsf{in}\ E_{4.4}$
$E_{4.4} = \mathsf{ret}\langle\!\langle f_1, C2\ \ (\lambda p''.\, \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle)\rangle\!\rangle$
$E_5 = \mathsf{bind}\, p' = \mathsf{store}()\ \mathsf{in}\ E_{5.1}$
$E_{5.1} = \mathsf{bind}\, x' = x\ p'\ \mathsf{in}\ E_{5.2}$
$E_{5.2} = \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x'\ \mathsf{in}\ E_{5.3}$
$E_{5.3} = \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = fm\ \mathsf{in}\ E_{5.4}$
$E_{5.4} = \mathsf{let}\langle\!\langle f_1, f_2\rangle\!\rangle = f\ \mathsf{in}\ E_{5.5}$
$E_{5.5} = \mathsf{ret}\langle\!\langle f_1, (C3\ \ (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle))\rangle\!\rangle$

$T_{0.0} = [3]\,\mathbf{1} \multimap \forall\alpha.Queue\ \alpha \multimap \mathbb{M}\,0\,(\alpha \otimes Queue\ \alpha)$
$T_{0.2} = [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{0.21} = \mathbb{M}\,0\,(\alpha \otimes Queue\ (\alpha \otimes \alpha))$
$T_{0.22} = (\alpha \otimes Queue\ (\alpha \otimes \alpha))$

$T_{0.23} = \mathit{Queue}\ (\alpha \otimes \alpha)$
$T_{0.3} = [0]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\alpha \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.31} = \mathbb{M}\,0\,((\alpha \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.32} = ((\alpha \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.33} = (\alpha \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.34} = \mathit{Queue}\ (\alpha \otimes \alpha)$
$T_{0.4} = [2]\,\mathbf{1} \multimap \mathbb{M}\,0\,((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.41} = \mathbb{M}\,0\,((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.411} = \mathbb{M}\,1\,((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.413} = \mathbb{M}\,3\,((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.42} = ((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.43} = (\alpha \otimes \alpha)$
$T_{0.44} = \mathit{Queue}\ (\alpha \otimes \alpha)$
$T_{0.5} = [1]\,\mathbf{1} \multimap \mathbb{M}\,0\,(((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.51} = \mathbb{M}\,0\,(((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.511} = \mathbb{M}\,1\,(((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.512} = \mathbb{M}\,2\,(((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.513} = \mathbb{M}\,3\,(((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.52} = (((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha)) \otimes \alpha)$
$T_{0.53} = ((\alpha \otimes \alpha) \otimes \mathit{Queue}\ (\alpha \otimes \alpha))$
$T_{0.54} = (\alpha \otimes \alpha)$
$T_{0.55} = \mathit{Queue}\ (\alpha \otimes \alpha)$
$T_0 = \mathbb{M}\,0\,(\alpha \otimes \mathit{Queue}\ \alpha)$
$T_1 = \mathbb{M}\,1\,(\alpha \otimes \mathit{Queue}\ \alpha)$
$T_2 = \mathbb{M}\,2\,(\alpha \otimes \mathit{Queue}\ \alpha)$

D5.51:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha, p'' : [0]\,\mathbf{1} \vdash \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle T_{0.31}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle) : T_{0.3}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash (\mathit{C3}\ (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle)) : \mathit{Queue}\ \alpha}$$

D5.5:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha \vdash f_1 : \alpha} \qquad D5.51}{\begin{array}{c}\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash \\ \langle\!\langle f_1, (\mathit{C3}\ (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle))\rangle\!\rangle : \alpha \otimes \mathit{Queue}\ \alpha\end{array}}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash \mathsf{ret}\langle\!\langle f_1, (\mathit{C3}\ (\lambda p''.\, \mathsf{ret}\langle\!\langle\langle\!\langle f_2, m\rangle\!\rangle, r\rangle\!\rangle))\rangle\!\rangle : T_1}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash E_{5.5} : T_1}$$

D5.4:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f : T_{0.54} \vdash f : T_{0.54}} \qquad D5.5}{\alpha; .; .; HT : T_{0.0}; f : T_{0.54}, m : T_{0.55}, r : \alpha \vdash \mathsf{let}\langle\!\langle f_1, f_2\rangle\!\rangle = f\ \mathsf{in}\ E_{5.5} : T_1}}{\alpha; .; .; HT : T_{0.0}; f : T_{0.54}, m : T_{0.55}, r : \alpha \vdash E_{5.4} : T_1}$$

D5.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.53} \vdash fm : T_{0.53}} \qquad D5.4}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.53}, r : \alpha \vdash \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = fm \;\mathsf{in}\; E_{5.4} : T_1}}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.53}, r : \alpha \vdash E_{5.3} : T_1}$$

D5.2:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash x' : T_{0.52}} \qquad D5.3}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x' \;\mathsf{in}\; E_{5.3} : T_1}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash E_{5.2} : T_1}$$

D5.1:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1]\,\mathbf{1} \vdash x\ p' : T_{0.51}} \qquad D5.2}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1]\,\mathbf{1} \vdash \mathsf{bind}\; x' = x\ p' \;\mathsf{in}\; E_{5.2} : T_1}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1]\,\mathbf{1} \vdash E_{5.1} : T_1}$$

D5:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,1\,([1]\,\mathbf{1})} \qquad D5.1}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5} \vdash E_5 : T_2}$$

D4.41:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44}, p'' : [1]\,\mathbf{1} \vdash \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle : T_{0.21}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44} \vdash (\lambda p''. \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle) : T_{0.2}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44} \vdash \mathit{C2}\ (\lambda p''. \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle) : \mathit{Queue}\ \alpha}$$

D4.4:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha \vdash f_1 : \alpha} \qquad D4.41}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash \langle\!\langle f_1, \mathit{C2}\ (\lambda p''. \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle)\rangle\!\rangle : \alpha \otimes \mathit{Queue}\ \alpha}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash \mathsf{ret}\langle\!\langle f_1, \mathit{C2}\ (\lambda p''. \mathsf{ret}\langle\!\langle f_2, m\rangle\!\rangle)\rangle\!\rangle : T_0}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash E_{4.4} : T_0}$$

D4.3:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f : T_{0.43} \vdash f : T_{0.43}} \qquad D4.4}{\alpha; .; .; HT : T_{0.0}; f : T_{0.43}, m : T_{0.44} \vdash \mathsf{let}\langle\!\langle f_1, f_2\rangle\!\rangle = f \;\mathsf{in}\; E_{4.4} : T_0}}{\alpha; .; .; HT : T_{0.0}; f : T_{0.43}, m : T_{0.44} \vdash E_{4.3} : T_0}$$

D4.2:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash x' : T_{0.42}} \qquad D4.3}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = x' \;\mathsf{in}\; E_{4.3} : T_0}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash E_{4.2} : T_0}$$

D4.1:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2]\,\mathbf{1} \vdash x\ p' : T_{0.41}} \qquad D4.2}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2]\,\mathbf{1} \vdash \mathsf{bind}\; x' = x\ p' \;\mathsf{in}\; E_{4.2} : T_0}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2]\,\mathbf{1} \vdash E_{4.1} : T_0}$$

D4:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,2\,[2]\,\mathbf{1}}\;D4.1}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4} \vdash E_4 : T_2}$$

D3.61:

$$\dfrac{}{\alpha; .; .; HT : T_{0.0}; r : \alpha, ht : T_{0.53} \vdash \mathsf{ret}\langle\langle ht, r\rangle\rangle : T_{0.51}}$$

D3.6:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3]\,\mathbf{1} \vdash HT\ p'''\ []\ m : \mathbb{M}\,0\,T_{0.53}}\;D3.61}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3]\,\mathbf{1} \vdash \mathsf{bind}\, ht = HT\ p'''\ []\ m\ \mathsf{in}\ \mathsf{ret}\langle\langle ht, r\rangle\rangle : T_{0.51}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3]\,\mathbf{1} \vdash E_{3.6} : T_{0.51}}$$

D3.5:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : [3]\,[3]\,\mathbf{1}}\;D3.6}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash \mathsf{bind}\, p''' = \mathsf{store}()\ \mathsf{in}\ E_{3.6} : T_{0.511}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash E_{3.5} : T_{0.513}}$$

D3.41:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; p'' : [1]\,\mathbf{1} \vdash p'' : [1]\,\mathbf{1}}\;D3.5}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p'' : [1]\,\mathbf{1} \vdash - = \mathsf{release}\, p''\ \mathsf{in}\ E_{3.5} : T_{0.512}}$$

D3.4:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; p_o : [2]\,\mathbf{1} \vdash p_o : [2]\,\mathbf{1}}\;D3.41}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p_o : [2]\,\mathbf{1}, p'' : [1]\,\mathbf{1} \vdash - = \mathsf{release}\, p_o\ \mathsf{in}\ E_{3.41} : T_{0.51}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p'' : [1]\,\mathbf{1} \vdash E_{3.4} : T_{0.51}}$$

D3.3:

$$\dfrac{\dfrac{\dfrac{D3.4}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash (\lambda p''.E_{3.4}) : T_{0.5}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash C5\ (\lambda p''.E_{3.4}) : Queue\ \alpha}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash E_{3.3} : Queue\ \alpha}$$

D3.2:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f : \alpha \vdash f : \alpha}\;D3.3}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash \langle\langle f, E_{3.3}\rangle\rangle : (\alpha \otimes Queue\ \alpha)}}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash \mathsf{ret}\langle\langle f, E_{3.3}\rangle\rangle : T_2}}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash E_{3.2} : T_2}$$

D3.12:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33} \vdash fm : T_{0.33}}\;D3.2}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33}, r : \alpha \vdash \mathsf{let}\langle\langle f, m\rangle\rangle = fm\ \mathsf{in}\ E_{3.2} : T_2}}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33}, r : \alpha \vdash E_{3.12} : T_2}$$

D3.11:

$$\frac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.32} \vdash x' : T_{0.32}} \qquad D3.12}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.32} \vdash \mathsf{let}\langle\!\langle fm, r\rangle\!\rangle = x' \;\mathsf{in}\; E_{3.12} : T_2}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.32} \vdash E_{3.11} : T_2}$$

D3.1:

$$\frac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3}, p' : [0]\,\mathbf{1} \vdash x\ p' : T_{0.31}} \qquad D3.11}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3}, p' : [0]\,\mathbf{1} \vdash \mathsf{bind}\, x' = x\ p' \;\mathsf{in}\; E_{3.11} : T_2}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3}, p' : [0]\,\mathbf{1}, p_o : [2]\,\mathbf{1} \vdash E_{3.1} : T_2}$$

D3.0:

$$\frac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3} \vdash \mathsf{store}() : \mathbb{M}\,2\,[2]\,\mathbf{1}} \qquad D3.1}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3}, p' : [0]\,\mathbf{1} \vdash \mathsf{bind}\, p_o = \mathsf{store}() \;\mathsf{in}\; E_{3.1} : T_2}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3}, p' : [0]\,\mathbf{1} \vdash E_{3.0} : T_2}$$

D3:

$$\frac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,0\,\mathbf{1}} \qquad D3.0}{\alpha; .; .; HT : T_{0.0}; x : T_{0.3} \vdash E_3 : T_2}$$

D2.51:

$$\frac{}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p_r : [3]\,\mathbf{1} \vdash HT\ p_r\ []\ m : T_{0.41}}$$

D2.5:

$$\frac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23} \vdash \mathsf{store}() : \mathbb{M}\,3\,[3]\,\mathbf{1}} \qquad D2.51}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23} \vdash \mathsf{bind}\, p_r = \mathsf{store}() \;\mathsf{in}\; HT\ p_r\ []\ m : T_{0.413}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23} \vdash E_{2.5} : T_{0.413}}$$

D2.4:

$$\frac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; p'' : [2]\,\mathbf{1} \vdash p'' : [2]\,\mathbf{1}} \qquad D2.5}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p'' : [2]\,\mathbf{1} \vdash \mathsf{release}\, p'' \;\mathsf{in}\; E_{2.5} : T_{0.411}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p'' : [2]\,\mathbf{1} \vdash E_{2.4} : T_{0.411}}$$

D2.3:

$$\frac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; p_o : [1]\,\mathbf{1} \vdash p_o : [1]\,\mathbf{1}} \qquad D2.4}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p_o : [1]\,\mathbf{1}, p'' : [2]\,\mathbf{1} \vdash - - = \mathsf{release}\, p_o \;\mathsf{in}\; E_{2.4} : T_{0.41}}$$

D2.21:

$$\frac{\dfrac{\dfrac{D2.3}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p_o : [1]\,\mathbf{1}, p'' : [2]\,\mathbf{1} \vdash E_{2.3} : T_{0.41}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p_o : [1]\,\mathbf{1} \vdash \lambda p''.E_{2.3} : T_{0.4}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.23}, p_o : [1]\,\mathbf{1} \vdash C4\ (\lambda p''.E_{2.3}) : Queue\ \alpha}$$

D2.2:

$$\frac{\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; f : \alpha \vdash f : \alpha} \qquad D2.21}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1]\,\mathbf{1} \vdash \langle\!\langle f, (C4\ (\lambda p''.E_{2.3}))\rangle\!\rangle : (\alpha \otimes Queue\ \alpha)}}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1]\,\mathbf{1} \vdash \mathsf{ret}\langle\!\langle f, (C4\ (\lambda p''.E_{2.3}))\rangle\!\rangle : T_0}}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1]\,\mathbf{1} \vdash E_{2.2} : T_0}$$

D2.11:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22} \vdash x' : T_{0.22}} \qquad D2.2}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22}, p_o : [1]\,\mathbf{1} \vdash \mathsf{let}\langle\!\langle f, m\rangle\!\rangle = x' \;\mathsf{in}\; E_{2.2} : T_0}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22}, p_o : [1]\,\mathbf{1} \vdash E_{2.11} : T_0}$$

D2.1:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1]\,\mathbf{1} \vdash x\ p' : T_{0.21}} \qquad D2.11}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p_o : [1]\,\mathbf{1}, p' : [1]\,\mathbf{1} \vdash \mathsf{bind}\, x' = x\ p' \;\mathsf{in}\; E_{2.11} : T_0}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p_o : [1]\,\mathbf{1}, p' : [1]\,\mathbf{1} \vdash E_{2.1} : T_0}$$

D2.0:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,1\,[1]\,\mathbf{1}} \qquad D2.1}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1]\,\mathbf{1} \vdash \mathsf{bind}\, p_o = \mathsf{store}() \;\mathsf{in}\; E_{2.1} : T_1}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1]\,\mathbf{1} \vdash E_{2.0} : T_1}$$

D2:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{store}() : \mathbb{M}\,1\,[1]\,\mathbf{1}} \qquad D2.0}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2} \vdash E_2 : T_2}$$

D1:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; x : \alpha \vdash \mathsf{ret}\ \langle\!\langle x, \mathit{C0}\rangle\!\rangle : T_2}}{\alpha; .; .; HT : T_{0.0}; x : \alpha \vdash E_1 : T_2}$$

D0:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \mathsf{fix}x.x : T_2}}{\alpha; .; .; HT : T_{0.0}; . \vdash E_0 : T_2}$$

D0.2:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; q : \mathit{Queue}\ \alpha \vdash q : \mathit{Queue}\ \alpha} \qquad D0 \qquad D1 \qquad D2 \qquad D3 \qquad D4 \qquad D5}{\alpha; .; .; HT : T_{0.0}; q : \mathit{Queue}\ \alpha \vdash E_{0.2} : T_2}$$

D0.1:

$$\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; . \vdash \uparrow^1 : \mathbb{M}\,1\,\mathbf{1}} \qquad D0.2}{\alpha; .; .; HT : T_{0.0}; q : \mathit{Queue}\ \alpha \vdash - = \uparrow^1; E_{0.2} : T_3}$$

Main derivation:

$$\dfrac{\dfrac{\dfrac{}{\alpha; .; .; HT : T_{0.0}; p : [3]\,\mathbf{1}, q : \mathit{Queue}\ \alpha \vdash p : [3]\,\mathbf{1}} \qquad D0.1}{\alpha; .; .; HT : T_{0.0}; p : [3]\,\mathbf{1}, q : \mathit{Queue}\ \alpha \vdash E_{0.1} : T_0}}{.; .; .; .; . \vdash E_{0.0} : T_{0.0}}$$

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